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Algebra Universalis

Well-foundedness conditions connected with left-distributivity

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ABSTRACT. We state a problem about free left-distributive algebras. If u and w are members of such an algebra write $u <_L w$ if $w = (((uu_0)u_1)\dots)u_n$ for some u_0, \dots, u_n . A conjecture about left-division in such algebras is given; it entails a normal form and that for every w the set of left divisors of w is well-ordered under $<_L$.

Let \mathcal{A}_k be the free left-distributive algebra on k -many generators, where k is a cardinal and the left-distributive law is the law $a(bc) = (ab)(ac)$. The purpose of this paper is to describe a conjecture connected with left division in \mathcal{A}_k , and some propositions related to it. This problem is one of a number of well-foundedness questions about free left-distributive algebras. Here are some basic facts (see [4] and [8], section 2 for respectively an account and a summary).

Fix a set S of cardinality k and let T be the set of S -terms in the language of one binary operation. For $\tau_0, \tau_1 \in T$, τ_1 is the result of a forward transformation on τ_0 if and only if τ_1 can be obtained from τ_0 by replacing a subterm of the form $a(bc)$ by $(ab)(ac)$. Then $\mathcal{A}_k = T/\equiv$, where $\tau \equiv \sigma$ iff σ can be obtained from τ by a sequence of forward transformations and/or their inverses. Write $[\tau]$ for the equivalence class of τ . Let $\tau_0 \rightarrow \tau_1$ mean that τ_1 is obtainable from τ_0 by a sequence of forward transformations. Then \mathcal{A}_k is confluent [1], that is, if $\tau_0 \equiv \tau_1$ then there's a τ such that $\tau_0 \rightarrow \tau$ and $\tau_1 \rightarrow \tau$. For $a, b \in \mathcal{A}_k$, write $a \mid b$ iff for some c , $ac = b$. Write $a <_L b$ if a is an iterated left divisor of b ; $b = ((aa_0)a_1)\dots a_n$ for some $n \geq 0$. Then $<_L$ is a partial ordering: transitivity is immediate, and irreflexivity ($a \not<_L a$), first shown in [6] as a corollary to a large cardinal axiom of set theory, was then shown in [3] without it—see [5] for a shorter proof of irreflexivity. And on $\mathcal{A} = \mathcal{A}_1$, $<_L$ is a linear ordering (modulo irreflexivity this was shown in different ways in [3] and [6]). On \mathcal{A}_k ($k > 1$), $<_L$ is not linear, but any linear ordering of S naturally induces a linear ordering of \mathcal{A}_k which extends $<_L$ [2]. Finally, \mathcal{A}_k satisfies left cancellation. For $k = 1$ this follows from the linearity of $<_L$ and the fact that

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$<_L$ is preserved under left translations, and for $k > 1$ it is derivable from the $k = 1$ case and confluence.

Example The linear order $<_L$ on \mathcal{A} is not a well-order. Let w be any member of \mathcal{A} of the form $r(st)$. Then

$$w = [((rs)r)(rs)][((rs)r)t]$$

and we have

$$((rs)r)(rs) <_L w.$$

The left hand side is of the same form, yielding an infinite descending sequence under $<_L$.

Definitions.

- (1) For $w \in \mathcal{A}_k$, $D_w = \{u \in \mathcal{A}_k : u \mid w\}$.
- (2) For $\langle a, b \rangle, \langle c, d \rangle \in \mathcal{A}_k \times \mathcal{A}_k$, $\langle c, d \rangle$ is an *LD-transformation* of $\langle a, b \rangle$ (and $\langle a, b \rangle$ is an *LD-inverse* of $\langle c, d \rangle$) if and only if for some r and s , $b = rs$, $c = ar$, and $d = as$ (so $ab = cd$).
- (3) A $\tau \in T$ is in *reduced normal form* just in case for every subterm $\sigma\gamma$ of τ , $D_{[\sigma]} \cap D_{[\gamma]} = \emptyset$.

In (2), $\langle u, v \rangle$ may have more than one LD transformation and LD inverse. These operations are more general than the forward transformations and their inverses at the term level: for example $\langle ab, (ac)d \rangle$ is an LD inverse of $\langle a(bc), (ab)d \rangle$, but the term $(\alpha\beta)((\alpha\gamma)\delta)$ is not obtainable from $(\alpha(\beta\gamma))((\alpha\beta)\delta)$ by the inverse of a forward transformation.

Conjecture.

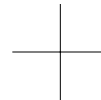
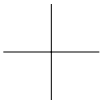
- (i) If $u, v, r, s \in \mathcal{A}_k$, $uv = rs$, $u \neq r$, then for one of u and r (say, u) there is a finite iteration of LD-transformations which takes $\langle u, v \rangle$ to $\langle r, s \rangle$.
- (ii) If $\langle r, s \rangle \in \mathcal{A}_k \times \mathcal{A}_k$ then any sequence $\langle r_0, s_0 \rangle, \dots, \langle r_i, s_i \rangle, \dots$ with $\langle r_0, s_0 \rangle = \langle r, s \rangle$, such that each $\langle r_{n+1}, s_{n+1} \rangle$ an LD-inverse of $\langle r_n, s_n \rangle$, is finite.
- (iii) If $w \in \mathcal{A}_k$ there is a unique term τ in reduced normal form with $[\tau] = w$.

Theorems on the existence of other normal forms have been proved in [6,7] and in [4].

Proposition 1. (a) *Conjecture (i) implies that for each $w \in \mathcal{A}_k$, D_w is linearly ordered by $<_L$.*

(b) *If Conjectures (i) and (ii) hold then for each $w \in \mathcal{A}_k$, D_w is well ordered by $<_L$.*

Proof. (a) Irreflexivity and transitivity follow from those properties of \mathcal{A}_k . And if $uv = rs = w$ with $u \neq r$, there is by (i) a sequence of LD-transformations taking, say, $\langle u, v \rangle$ to $\langle r, s \rangle$, which would witness $u <_L r$.



(b) Suppose there is an infinite sequence $u_0 >_L u_1 > \dots >_L u_n >_L \dots$ of members of D_w . Writing $u_n v_n = w$, there is by (i) a sequence of LD-inverse operations which take $\langle u_n, v_n \rangle$ to $\langle u_{n+1}, v_{n+1} \rangle$, for each n . This infinite iteration of LD-inverses contradicts (ii). \square

We briefly consider some other hypotheses about \mathcal{A}_k and their connections with one another. Some statements are explicitly mentioned because they have a plausibility argument, namely they have been proved to be true in \mathcal{A} (see remarks at the end).

Proposition 2. *Suppose the following condition holds:*

(*) *For all $a, b, c \in \mathcal{A}_k$, if $a \mid b$ and $a \mid bc$ then $a \mid c$.*

Then, for any $\langle u, v \rangle, \langle r, s \rangle \in \mathcal{A}_k \times \mathcal{A}_k$ there is a finite sequence of LD transformations which takes $\langle u, v \rangle$ to $\langle r, s \rangle$ if and only if $uv = rs$ and there is a sequence $u_i (0 \leq i \leq n)$, with $u_0 = u$, $u_i \mid u_{i+1} (i < n)$, and $u_n = r$, such that each $u_i \mid rs$.

Proof. For the right to left direction, write $rs = u_i t_i$, $u_{i+1} = u_i q_i$. Since $u_i \mid u_{i+1}$ and $u_i \mid u_{i+1} t_{i+1}$, $u_i s_i = t_{i+1}$ for some s_i , by (*). Then $\langle u_{i+1}, t_{i+1} \rangle = \langle u_i s_i, u_i q_i \rangle$ has an LD-inverse $\langle u_i, q_i s_i \rangle$, and $q_i s_i = t_i$ by left cancellation. Also by left cancellataion $t_0 = v$ and $t_n = s$. This gives a sequence of LD inverses taking $\langle r, s \rangle$ to $\langle u, v \rangle$. \square

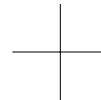
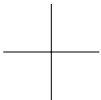
Proposition 3. *Conjecture (i) implies that the reduced normal form of a $w \in \mathcal{A}_k$ is unique if it exists.*

Lemma 1. *Assume (i). Then if $u, v, r, s \in \mathcal{A}_k$, $uv = rs$, and $D_u \cap D_v = D_r \cap D_s = \emptyset$, then $u = r$ and $v = s$.*

Proof. Let $w = uv = rs$. It suffices by left cancellation to show $u = r$. Suppose $u \neq r$. Then by (i), for one of u and r , say r , we have that $\langle r, s \rangle$ is in the range of an LD-transformation. Thus $D_r \cap D_s \neq \emptyset$, a contradiction. \square

To show uniqueness of reduced normal forms, we show by induction on length τ ($\tau \in T$), that if $\tau \equiv \sigma$ and τ, σ are in reduced normal form, then $\tau = \sigma$. We are done if either $[\tau]$ or $[\sigma]$ is a generator, so assume $\tau = \tau_0 \tau_1$, $\sigma = \sigma_0 \sigma_1$. By the normal form, $D_{[\tau_0]} \cap D_{[\tau_1]} = D_{[\sigma_0]} \cap D_{[\sigma_1]} = \emptyset$. By the lemma then, $[\tau_0] = [\sigma_0]$ and $[\tau_1] = [\sigma_1]$. Since τ_0, τ_1 have smaller length than τ and the τ_i 's and σ_i 's are in reduced normal form, $\tau_i = \sigma_i$ holds by induction. Thus $\tau = \sigma$, proving Proposition 3.

Assuming the conjecture, part (iii) is expressible as follows. For every $w \in \mathcal{A}_k$, let T_w be the \mathcal{A}_k -labelled tree, with root node labelled by w , such that nodes which are labelled by generators are terminal, and every node labelled by a nongenerator v has two immediate successors which are labelled by the elements r and s such that $rs = v$ and r is the $<_L$ -least member of D_v . Then for each w , T_w is finite.



A number of the above statements are known in the one-generator case (results of the first author). Namely, if $w \in \mathcal{A}$, then Conjecture (i) is true for w . It follows that the conclusion of Lemma 1 is true for \mathcal{A} , and that the reduced normal form of a member of \mathcal{A} is unique if it exists. Also, (*) is true for \mathcal{A} . Conjectures (ii) and (iii) are open for \mathcal{A} .

A number of theorems about free left distributive algebras have related versions about the braid groups, and vice-versa (see [3] and [4], also see [8], Section 3 for facts about braids which are related to the question of well ordering of D_w under $<_L$).

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