Natural Gorensteinization of toric varieties (4 November 2008)

One purpose of this note is to suggest what should be a natural Gorensteinization process for toric varieties. A deeper purpose is that convergence of the Gorensteinization process is proven to correspond to finite type of a torus equivariant subsheaf of graded algebras of the graded Grauert-Riemenshneider sheaf, which still is large enough to contain the sheaf R which determines convergence of the Nash process. Even for a toric variety known to be Gorenstein the sheaf we describe is strictly smaller than ω in general.

Let V_0 be a toric variety. Then V_0 contains a copy of a split n torus T whose is complement the anticanonical divisor with irreducible components $D_{i_1}, ..., D_{i_s}$, each D_α defined by vanishing of a primitive character $\chi_\alpha \in X(T)$. The dual character group $\widehat{X(T)}$ is thereby partitioned into a disjoint union consisting of the trivial element together with open facets given by systems of equations

$$\begin{cases} v(\chi_{i_1}) > 0 \\ \quad ... \\ v(\chi_{i_m}) > 0 \end{cases}$$

corresponding to particular sequences $i_1, ..., i_m$, namely those sequences such that $D_{i_1} \cap ... \cap D_{i_m}$ is a minimal intersection comprising a single point. Choosing an orientation on $\mathbb{G}_m$ and associated embedding $\mathbb{G}_m \subset \mathbb{A}^1$ the dual characters in the open cone are those maps $\mathbb{G}_m \to T$ which extend to maps $\mathbb{A}^1 \to V_0$ sending the origin to this particular point.

The closed facet C associated to each open facet is defined by

$$\begin{cases} v(\chi_{i_1}) \geq 0 \\ \quad ... \\ v(\chi_{i_m}) \geq 0 \end{cases}$$

and its dual $\widehat{C}$ is the set of characters χ such that $v(\chi) \geq 0$ for $v \in C$. These are the characters $T \to \mathbb{G}_m$ which when viewed as functions to $\mathbb{A}^1$ via the inclusion $\mathbb{G}_m \to \mathbb{A}^1$ extend to the variety V in which all components of the anticanonical divisor *not including* $D_{i_1}, ..., D_{i_m}$ have been deleted. In other words if we write the canonical divisor $K = K_1 + K_2$ where the effective divisor $-K_1$ is supported on $D_{i_1}, ..., D_{i_m}$ and the effective divisor $-K_2$ is prime to

$-K_1$, then $\widehat{C}$ is the set of characters belonging to the union over natural numbers N of $\Gamma(V_0, \mathcal{O}_{V_0}(-NK_2))$.

Define the open dual facet associated to C to be the set of characters χ such that

$$v(\chi) > 0 \text{ for } \chi \in C.$$

These are the characters well defined zero at the generic points of $D_{i_1}, ..., D_{i_m}$. Note that the divisors K_1 and K_2 are reduced. The characters in the open cone are those in the union over natural numbers N of $\Gamma(V_0, \mathcal{O}_{V_0}(NK_2 - K_1))$. The characters which when viewed as functions to $\mathbb{A}^1$ have unrestricted poles on Y_2 and at least simple zeroes on Y_1.

Fix a choice of C. Associated now to each *rational* character $\chi \in X(T) \otimes \mathbb{Q}$ let C_χ be the set of dual characters v such that $v(\chi)$ equals the minimum value of v upon all the points (integral characters) in the the open dual facet of C. It is possible that $C_\chi = C'_\chi$ when χ and χ' are distinct elements of the open dual facet associated to C.

Remark. An alternative definition of C_χ is to start with the set $\widehat{C}$ of lattice points, remove those on the boundary of the convex hull, then consider the lattice polyhedron which is the convex hull of the remaining points. If χ does not belong to this convex hull then $C\chi$ is empty. If it belongs, then C_χ is dual to the largest facet of the polyhedron which contains χ. Thus in the theorem below, we either shall view the C_χ as a set of facets rather than a family, or we shall choose one χ in the interior of each facet of the polyhedron spanned by the nonboundary lattice points of $\widehat{C}$.

Theorem.

1. The facets C_χ subdivide the C into a fan. Therefore they describe a variety V_1 and a proper birational morphism $V_1 \to V_0$.

2. If the sequence $...V_2 \to V_1 \to V_0$ stabilizes then V_i is Gorenstein for large i.

3. If the cones are simplicial for large i then the sequence $...V_2 \to V_1 \to V_0$ does indeed stabilize.

4. No example is yet known where it does not stabilize.

Proof. The convergence if all cones were simplicial is a volume calculation, based upon performing an elementary transformation. Namely, if the boundary of the convex hull of a simplical cone lattice is removed, then the cones at the new vertices are of two types: those with all new edges, where the affine span of the vertex and primitive edge rays is contained in the original, and those which have a new edge. These need an elementary transformation applied before they can be contained in the original hull. In both cases the volume decreases as a proportion of the volume of the lattice, providing a well-ordered decreasing invariant.

Remark. Each map $V_{i+1} \to V_i$ is the result of blowing up $\mathcal{O}_{V_i}(K_{V_i})$ as a rank one coherent sheaf.

Example. Consider the cone in the $\widehat{X}(T)$ spanned by vectors with coordinates

$$\{(0,3,1),(0,-1,7),(8,3,-1)\}.$$

In the first step of the process this is subdivided such that the two dimensional cone spanned by

$$\{(0,3,1),(0,-1,7)\}$$

is split into two coplanar two-dimensional cones spanned by

$$\{(0,3,1),(0,0,1)\}$$

and

$$\{(0,0,1),(0,-1,7)\}$$

while the three dimensional cone is split into the union of three nonsimplicial cones with spanning sets

$$\{(2,1,0,),(5,2,0),(3,3,1),(0,0,1)\},$$

$$\{(2,1,0),(3,3,1),(0,3,1),(8,3,-1)\},$$

$$\{(5,2,0),(0,0,1),(0,-1,7),(8,3,-1)\}$$

and two simplicial cones, which are already Gorenstein, with spanning sets

$$\{(8,3,-1),(0,3,1),(0,-1,7)\},$$

$$\{(2,1,0),(5,2,0),(8,3,-1)\}$$

In the second step, which is the final step, the three nonsimplicial Gorenstein cones are each divided into two simplicial Gorenstein cones by inserting two-dimensional facets with spanning sets respectively

$$\{(0,0,1),(2,1,0)\}.$$

$$\{(0,3,1),(2,1,0)\}.$$

$$\{(0,0,1),(8,3,-1)\},$$

The reverse of this second step is a small contraction.

It is reasonable to ask whether this process always finishes in finitely many steps. However we do know,

Corollary. Since in dimension two all cones are simplicial, the process converges for toric surfaces.

Later we will look for convergence conditions for cones not necessarily simplicial. First we do something more important and relate convergence to finite type of a T-invariant sheaf of graded algebras. For any convex set S of characters write ∂S for those on the boundary of the convex hull, and we write

$$\delta(S) = S \setminus \partial(S).$$

Lemma. For convex lattice polyhedra A, B,

1. $A\delta(B) \subset \delta(AB)$
2. $\delta(A)^2 \subset \delta(A\delta A)$.
3. If A is a cone then for $i, j \geq 0$ $\delta^i(A)\delta^j A \subset A\delta^{i+j}A$.
4. $B^{i-1}\delta(AB) = \delta(AB^i)$.

Proof. For the first part, note $A\delta B \subset \delta(AB)$ as if we are given a product ab with b not on the boundary of the convex hull of B then b is an interior point of a simplex with lattice vertices $x_1, ..., x_n$ in B then ab is also the interior point of a simplex with lattice vertices

$ax_1, ..., ax_n$ in AB. This means, if ab can be rewritten with either a or b in the interior, ab is in the interior.

For the second part, let $b = \delta(A)$.

For the third part, we have

$$\delta^i(A)\delta^j(A) \subset \delta^{i+j}(A^2)$$

and since A is a cone – although multiplication is not well defined without choice of an origin – we can choose the vertex to be an origin, and then we may write

$$A^2 = A$$

$$\delta^{i+j}(A)A \subset \delta^{i+j}(A).$$

Then the desired formula follows, and note that the inclusion between right and left sides of the final result are well defined without reference to any origin. For the last part, we use the fact that near a point of the boundary of $\delta(I^N)$ the convex hull of I^N looks like the convex hull of I translated by a point of I^{N-1}. QED.

For each cone C of V let $\widehat{C}$ be the dual and let

$$E_0 = \delta\widehat{C}$$

$$E_{i+1} = \delta(E_0 E_1 ... E_i)$$

for $i = 0, 1, 2, ...$

Then by the previous lemma,

$$E_i^2 \subset \delta(E_0 ... E_{i-1} \cdot \delta(E_0 ... E_{i-1})) = E_{i+1}$$

and so there is a graded monomial algebra whose terms are ideals of $\widehat{C}$ which is generated in degree 2^i by $\delta(\widehat{C}E_0E_1...E_{i-1})$. We will see that convergence of the Gorensteinization process is equivalent to finite type of the graded algebra for each cone C.

The term $\widehat{C}$ does not affect the answer, but it is put there so that the product can be described without reference to an origin; it makes sense if the lattice is viewed as an affine lattice only.

Remark – patching The question of patching together the affine parts into a sheaf of graded algebras comes down to the question of whether we may apply δ to a sheaf of T invariant ideals and obtain another. We may work locally on one cone C and a wall of C gives a pair of dual cones which we may consider to be part of a polyhedron with two vertices, and we may consider that the polyhedron is a lattice polyhedron which is the intersection of two lattice cones, and we ask the question whether a region along the edge would look the same if we remove points in the boundary of the convex hull of one cone, versus the other. Now, this is poorly described but true, that the situation we are looking at, dual to a pair of cones meeting along a wall, is a situation where a number of hyperplanes meet along a line. These are the duals of the one-facets which span the wall of intersection of the cones. These hyperplanes fit together to make part of the boundary of the polyhedron, and the point is that in a neighbourhood of a point on the line segment which is between, but not including, the two vertices of the dual cones, the boundary of the polyhedron can be considered to be built by walls of either dual cone, and it does not matter which dual cone we consider when removing then the lattice points in the boundary of the convex hull.

This proves then that the E_i and the graded algebra may be considered to be sheaves of ideals, and a sheaf of graded algebras. Thus we have constructed a sheaf of graded algebras whose finite type is equivalent to convergence of the Gorensteinization. And, this sheaf of graded algebras is contained in the graded Grauert-Riemenshneider sheaf ω; and yet it contains the sheaf R whose finite type is equivalent to convergence of the Nash process.

Corollary. Let V_0 be a toric variety and let $\pi : V_1 \to V_0$ be blowing up $\mathcal{O}_{V_0}(K_{V_0})$. Let $E = -\pi^* K_{V_0}$ which is an effective Cartier divisor. Then $K_{V_1} - E$ is relatively basepoint free.

Corollary. If we call K_i the divisor K_{V_i} pulled back to V_j for $j > i$ then the divisor

$$D_i = K_i + (K_{i-1} + 2K_{i-2} + ... + 2^{i-1} K_0)$$

is relatively basepoint free over V_0. The sum $D_1 + ... + D_i$ is relatively very ample.

Of course the divisors D_i can be defined for non toric variieties, I do not know if they are still relatively basepoint free or if the sum is relatively very ample.

The sheaf of graded algebras is generated by the pushforward of $\mathcal{O}(D_i)$ in degree 2^i.

Theorem. The graded sheaf is finite type if and only if $V_{i+1} \to V_i$ is an isomorphism for large i.

Proof. Since V_{i+1} is the blowup of $E_0....E_i$ it suffices to show that when E_{i+1} is a divisor of a power of $E_0....E_i$ for large i then $E_{i+1} = E_i^2$ for large i.

In an analogous situation, considering Nash blowing up, we first worked by hand using derivations and later realized that the result could be deduced by interpreting the morphism there $L_i^{r+2} \to L_{i+1}$ as one which pulls back up to twisting, to the map which determines ramification of the Nash blowup. A similar geometric proof is likely to be possible here too, however it is also easy to work directly combinatorially. Thus we deduce what we want from this lemma.

Lemma. Let I be the lattice points in a lattice polyhedron, and $N > 0$ a natural number. Suppose for some other set of points in a convex lattice polyhedron X

$$X\delta(I) = I^N.$$

Then $\delta(I\delta(I)) = \delta(I)^2$.

Proof.

$$\begin{aligned} X\delta(I\delta(I)) &\subset \delta(XI\delta(I)) \\ &= \delta(I^{N+1}) \\ &= I^N\delta(I) \\ &= X\delta(I)^2 \end{aligned}$$

and the result follows by convexity.

Now, taking $I = E_0...E_i$ we see that when it is the case that $V_{i+2} \to V_{i+1}$ is an isomorphism, then there is some X such that $XE_{i+1} = (E_0....E_i)^N$ for some N. Then $X\delta(I) = I^N$ and the lemma gives

$$E_{i+2} = \delta(I\delta) = \delta(I)^2 = E_{i+1}^2.$$

Now we can focus on trying to see what it means for the inclusion $\delta(I)^2 \subset \delta(I\delta(I))$ to be equality. Note that $I\delta(I)$ is a union of translates of I (one for each lattice point of $\delta(I)$) and the issue is whether there is any interior point of the union which is not an interior point of any one of the sepearate translates.

Here is an interesting lemma.

Lemma. Suppose A and B are lattice polyhedra. Let A' be the set of inverses of elements of A. Suppose that for any translate of A' which meets B only at boundary points of both there is a lattice hyperplane such that B is in one closed halfspace and the translate of A in the other. Then $\delta(AB) = A\delta(B) \cup B\delta(A)$.

The proof is that a point x of AB which can only be expressed as a product of two boundary points correponds to an origin such that B meets $-A$ only at boundary points. Then B and A are on the same side of a hyperplane containing x and so is BA

If this hypothesis holds for all $\delta^i(\widehat{C})$, which is extremely likely to be true, then we may calculate the E_i by 'multiplying out.' For instance

$$E_1 = \delta(E_0\delta(E_0)) = \delta(E_0)^2 \cup E_0\delta^2 E_0,$$

$$E_2 = (\delta(E_0))^4 \cup E_0(\delta(E_0))^2)\delta^2 E_0 \cup E_0(\delta(E_0))^2\delta^2(E_0) \cup E_0^2(\delta^2(E_0))^2 \cup E_0^2\delta(E_0)\delta^3(E_0),$$

and in general we obtain that E_j is the union of all monomials in the $\delta^s(E_0)$ whose degree is equal to 2^j in two different senses, namely if $[\delta^s(E_0)]^t$ is given degree t, or degree st.

The number of terms is then one less than the number of partitions of 2^j since it is the number of simultaneous positive (or zero) integer solutions of

$$d_0 + d_1 + ... + d_{2^j-1} = 2^j$$

$$0d_0 + 1d_1 + 2d_2 + ... + (2^j - 1)d_{2^j-1} = 2^j.$$

Here d_0 is determined by the other variables by the first equation, and the second equation says the remaining d_i describe a partition of 2^j into more than one part. The solution $(d_0, d_1, ...)$ gives the term

$$(E_0)^{d_0}(\delta(E_0))^{d_1}\delta^2(E_0)^{d_2}....$$

If it is the case that

$$\delta^i(E_0)\delta^j(E_0) \subset \delta^{i-1}(E_0)\delta^{j+1}(E_0)$$

for large $i > j$ then the solution $(d_0, d_1, ...) = (2^{j-1}, 2^{j-2}, ..., 2, 1, 1)$ includes all others and in this case by convexity finite type is equivalent to the inclusion $\delta^{i+1}(E_0)\delta^{i-1}(E_0) \subset \delta^i(E_0)^2$ being equality for large i. Also in this case because there is only one term, if it is the case that in addition to these hypotheses the graded algebra with $\delta^i(\widehat{C})$ in degree i is finite type, then it would be the case that a finite set of the E_i would dominate all the others and in this case too the algebra generated by E_i in degree 2^i would be finite type.

Relation with R

The definition of the graded sheaf E can be changed analogous to the change we made in passing from $n+2$ to $n+3$ for the graded sheaf R. Namely instead of defining $E_{i+1} = \delta(\widehat{C}E_0...E_i)$ we could define $E_{i+1} = (\widehat{C}E_0...E_i)^n\delta(\widehat{C}E_0...E_i)$ and then this smaller graded sheaf still contains R.

Let's discuss this. What takes place is that this modified construction gives a graded sheaf whose $Proj$ rather than only deciding convergence of Gorensteinization by whether it is finite type or not – although it still does have this property – a Ctually converges to the Gorensteinization when it exists, and it is a smaller graded sheaf which still contains R and is only slightly larger than R. In the case of A_2 it is I believe the inclusion of R in this sheaf which becomes an equality in high degrees.

Here we have been a bit pedestrian talking about δ as operating on lattice polyhedra. Our F had been a functor, but for a coherent subsheaf $\mathcal{F} \subset \mathcal{O}_(H)$ we had $F(\mathcal{F}) \subset \mathcal{O}_V(K + (n+1)H)$. In the toric case working locally we consider a single dual cone $\widehat{C}$. A torus equivariant Weil divisor H assigns an integer to each component of the anticanonical divisor, ie each generator of the original fan, and tells us what is the required valuation a character must have on each component. Correspondingly there is a unique way to translate the boundary hyperplanes of $\widehat{C}$, such that the modified dual cone contains the characters which are global sections on $Spec(\widehat{C})$ of $\mathcal{O}_V(H)$.

If $\mathcal{F}$ is contained in $\mathcal{O}(H)$ a torus equivariant coherent integrally closed subsheaf, it gives us a convex lattice polyhedron in each dual cone so modified by translating its boundary hyperplanes.

Rightfully speaking, when we apply F the result should not naturally be viewed as giving us anymore a family of lattice polyhedra in the dual cones, but rather within the $\mathcal{O}((n+1)H + K)$. It is fairly inessential whether we decide to multiply these sections all by $dx_1dx_2....dx_n/(x_1x_2...x_n)$ for x_i a transcendence base of characters, as this is expressing the global isomorphism between $\mathcal{O}(K)$ and differential n forms with primary components of codimension ≥ 2 discarded.

The fact that the sections of $\mathcal{F}$ on $Spec(\widehat{C})$ are contained in the cone corresponding to H implies the products of $n+1$ such characterS are contained in the cone correspnding to $(n+1)H$, in fact I think if $\mathcal{F} = \mathcal{O}(H)$ we get this exactly, and then the extra factor of K is allowed because when the characters are required to be affinely independent the result cannot actually lie on any one of the boundary hyperplanes. In fact, the global sections of $\mathcal{O}((n+1)H+K)$ on $Spec(\widehat{C})$ comprise the lattice polyhedron which is the convex hull of the lattice points in the interior of the polyhedron corresponding to $\mathcal{O}(H)$ itself.

Now, I am going to assert that multiplying a lattice polyhedron by $(n+1)$ and taking set of lattice points in the interior is no different than multiplying by 2, taking lattice points in the interior, and then multiplying by $(n-1)$ times the latice points in the original polyhedron.

In other words, I claim

Lemma. For P a convex lattice polyhedron,

$$\delta(P^{n+1}) = P^{n-1}\delta(P^2).$$

Proof. This follows from part 4. of the earlier lemma, taking $A = B$.

So what we see is that the sheaf E redefined this way, although it is smaller now and also defines a higher Proj sheaf which dominates the earlier one, it now is only slightly larger than R.

What we have in this situation is a very concrete realization of the reason Nash blowups converge for any toric surface where eventually the $\Lambda^2\Omega_{V_i}/torsion$ become reflexive; because the L_i are then exactly equal to the E_i and the graded sheaf which determines finite type of the Nash blowups is the same one which not only now determines finite type of the Gorenstienization, but actually has that the Gorensteinization is Proj of that graded sheaf.

In other words, in the toric case in dimension n when eventually the $\Lambda^n\Omega_{V_i}/torsion$ are reflexive, then the graded sheaf such that pulling back along its Proj converts the penultimate term of the Nash tower to the ultimate term when it is finite type, is also equal to the graded sheaf whose finite type is equivalent to convergence of Gorensteinization by repeatedly blowing up the defining ideal of the free torus orbit, and so when in addition the Gorensteinization process finishes, as is always the case in dimension 2, then the Gorensteinization which results is precisely such a variety which when pulled back to an appropriate term of the Nash tower, results in a completion of the Nash tower, ie a nonsingular variety which is the Nash blowup of that term.

We see in this situation though that the Gorensteinization is exactly the last term of the tower. We have one process known to converge for toric surfaces, the Gorensteinization, and another known to give a desingularization when it converges, the Nash process, and so it is just clear that the hypothesis that the two processes agree would lead trivially logically to the conclusion that the Nash process is a convergent desingularization. There is one extra conclusion to draw in such a case; namely that Proj(R) which is not always nonsingular, is here nonsingular, and agrees with the desingularization.

Yet, if it is only true that *for sufficiently large* i the inclusion $L_i \subset E_i$ is equality, then the Gorensteinization is an interesting auxiliary variety which pulls back to create the Nash resolution. When one has reached the next-to-last stage of the Nash tower, one reaches a variety which fails nonsingularity only because it fails Gorenstein. This is a variety whose Gorensteinization is its one-step Nash desingularization.

We could also ask when there is an induced morphism only, necessarily affine, and then we are asking when it is the case that the radical ideal in the sheaf generated by the E_i which is generated by the L_i is the unit ideal.

In other words, start with a variety where the Gorensteinization is known to converge; we look at the subscheme in the Gorensteinization variety generated by the ideal sheaves L_i, and as we allow more and more L_i we have a finite descending chain of closed subschemes of the Gorensteinization. This finishes in finitely many steps, no matter what; and if it finishes by being empty then this means that the Gorensteinization dominates the auxiliary Nash variety, and then the auxiliary variety is finite type and therefore the Nash resolution converges.

For example, if we start with a variety already known to be Gorenstein, then the L_i define a sheaf of ideals in the variety itself. They are embedded as sheaves of ideals by twisting by the appropriate invertible sheaves E_i^{-1}. When this is empty for large i what is happening is that the $L_i = F(L_0 ... L_{i-1})$ are actually invertible sheaves on V for large i. So the reason in this case that the Nash process converges is because high L_i are actually invertible on V for large i.

This cannot happen unless V is already nonsingular. For, $\mathcal{P}roj(R)$ has as its truncation the graded sheaf of powers of L_i for some large i, and this is giving $\mathcal{P}roj(R) \to V$ an isomorphism. But pulling back along $\mathcal{P}roj(R)$ converts the next-to-last term of the Nash tower of V to the last term.

So for a variety which is already Gorenstein it just cannot happen that $L_i \subset E_i$ is equality for large i.

We could ask, though, could it happen for a variety which is not Gorenstein that $L_i \subset E_i$ is an isomorphism for large i. We could ask, can we deduce the same contradiction by pulling back to the Proj of the sheaf generated by all the E_i. Because in the definition of the E_i we have E_i is divisible by $E_0, ..., E_{i-1}$ (unlike in the definition of the L_i) this variety is the same as blowing up any one E_i, for one sufficiently large i, and then it is the same as blowing up one L_i, ie it is $\mathcal{P}roj(R)$.

So we can look on $\mathcal{P}roj(R)$, and say here is a Gorenstein variety and the L_i with a shift of i pull back to the L_i for this variety, and therefore they are all invertible here, and so $\mathcal{P}roj(R)$ must by our earlier argument be nonsingular.

In other words, our original variety V was a variety with a nonsingular Gorensteinization.

So if we find a variety for which $L_i \subset E_i$ is an isomorphism for large i then we have found a variety for which the Gorensteinization is nonsingular. It is indeed not the same as the Nash resolution, which also exists and is higher. Here, the pullback of the next-to-last term of the Nash tower with the Gorensteinization of V is the Nash resolution, and there is an induced map of nonsingular varieties from the Nash resolution of V to the nonsingular Gorensteinization of V.

This stragegy then, of proving $L_i \subset E_i$ is an isomorphism for large i, would do nothing in case of a singular Gorenstein variety, or a variety whose natural Gorensteinization is singular. What it does do is provide a sufficient condition for an easier resolution strategy to exist, of blowing up the canonical divisor as closed scheme repeatedly until a nonsingular variety is reached. It is a nonsingularity criterion for the Gorensteinization which has as a side-effect that when it is true and when the Gorensteinization exists, it implies also that the Nash resolution exists also, and there is a natural map of nonsingular varieties from the Nash resolution to the Gorensteinizatoin.

A useful generalization then would be to dispense with the L_i and instead consider the natural map

$$F(\widehat{C}E_0 ... E_{i-1}) \subset E_i.$$

This is a property of the E_i equialent to saying we can discard a finite beginning portion of the Gorensteinization tower and achieve all the above later on. In other words, there is no reason to keep track of the earlier L_i.

This hypothesis implies existence of a desingularization that is different than the Nash desingularization, and if this holds eventually then eventually all the above is taking place. In other words eventually there is a term in the Gorensteinization tower with convergent Nash blowup, but since we are not keeping track of when this occurs, this is a trivial statement as later on in the Gorensteinization tower is a variety which is actually nonsingular.

In other words, keeping track of the L_i was keeping track of at what point in the Gorensteinization tower we might expect to realize a variety with a finite Nash tower, but anyway it is maybe better to forget about the Nash towers if our hypothesis is so strong as to provide a nonsingular Gorensteinization.

And so we have a theorem whose hypotheses are certainly not generically true

Theorem. With E_i defined as above, if for large i the inclusions

$$F(\widehat{C}E_0 ... E_{i-1}) \subset E_i$$

are equality and and $Bl_{E_i}V$ is Gorenstein, then it is nonsingular.

There is a reasonable likelihood of extending the definition of the E_i to general varieties (not necessarily toric) such that the statement of the theorem continues to hold. But note that the hypotheses are never true for a singular Gorenstein variety, so this is a theorem about varieties whose failure of nonsingularity is caused by failure of Gorenstein.

Note that we can give a direct proof of the theorem, and see precisely what hypotheses are necessary, by the theorem in 'Finite Generation and the Gauss process' which said that a necessary and sufficient condition of resolvability is a torsion free coherent sheaf of rank one J such that $F(J)^{n+2} = F(JF(J))$. Here we take $\widehat{C}E_0...E_{i-1} = J$ and the condition $E_i = F(J)$ gives

$$F(J)^{r+2} = E_i^{r+2} = E_{i+1} = F(\widehat{C}E_0...E_i)$$
$$= F(\widehat{C}E_0...E_{i-1}F(\widehat{C}E_0...E_{i-1}))$$
$$= F(JF(J).$$

The particular values of i for which various things need to hold can be verified then by unwinding the proof more carefully, but certainly it is only necessary to verify the condition for two or three values of i.

And this strategy too has an easy description and explanation. It says look at the sequence of sheaves which give Gorensteinization, and check whether they eventually are the same sheaves which give Nash blowing up of the previous step of the Gorensteinization process. Of course it is obvious that if they are the same, the Gorensteinization process has provided a desingularization.

Then, having found a pair of sheaves and an inclusion, we have an explicit condition to check ie an explicitation of a very obvious strategy, but one which could only work when one is lucky enough that the canonical Gorenstienization leads to something nonsingular.

The condition is the one we have constantly encountered, and it may be considered a condition on the dual cones during Gorensteinization, that eventually any interior character is a product of $n + 1$ characters in affine general position. The conjunction of this and the condition that the set of such characters is just multiples of one principal generator times all the characters in the cone (the Gorenstein condition) implies nonsingularity of course. During Gorensteinization one does expect to eventually reach that there exists this principal character, and the new idea here is to separately and independently watch for the set of interior characters to become products of $n + 1$ characters in general position. This will not always take place.

To put things in context algebraically, there is the filtration of ω by various graded subsheaves, and here we are considering the case that the integral closure of R is one of the subsheaves which we are calling E related to Gorensteinization.

We have yet to consider things like pushing forwards sheaves R.

It is certainly the case that in any tower of higher and higher blowups of V the increasing filtration of ω the graded Grauert-Riemenschneider sheaf, by the direct images of the R sheaves on the terms of the tower, is a filtration that must converge in finitely many steps if it converges at all, due to the fact ω is known to be finite type. And conversely it must converge to ω if the tower leads to a desingularization. Here we have kept within a fairly small sheaf E .