

Arc lifting for the Nash manifold

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Preface

Our understanding of nature doesn't do anything like encompass nature.

In 'The olden times' Ian Hislop [1] claims that Tolkien's romanticization of nature is misleading, that the home town which is implicitly the real setting of his fantasies had already been industrialized before he lived there. What Tolkien writes about actually does make sense. Not when you go to his town as an investigator, it is when you think of stories over the generations.

My Mom once said she'd lived in a mill-town on a river, with its 'mill-pond' downstream from the water-wheel. The town was at the edge of what they called the 'never-ending forest.' You could see Northern lights at night, and the people living there, immigrants from Norway, knew, loved and understood the trees, mosses and types of fungus shapes that grow on healthy trees. Also the types of 'granddaddy long-legs' spiders, salamanders, and all types of living things. These live on one side of the never-ending forest, the side which will never be completely cut down. As the boundary moved northwards, the goodness of the never-ending forest was absorbed into their lives.

Context of the Nash-Semple question

Nash's question in n dimensions is this: each time the indeterminacy of n forms is resolved, the regular locus of the resolution is an open subset which extends the regular locus of a singular manifold, yielding a topological space with an open cover by a countable ascending series of open manifolds.

As a question in the classical topology, it is whether a continuous path starting in the original manifold, and whose whole image there is compact, also has an endpoint in one of the open subsets whose union is the Nash manifold, or, whether it can actually go on forever.

Resolving the indeterminacy of n -forms starts with thinking about smoothly parametrized paths in space. The image of a path retains some aspects of an implicit parametrization. An image in projective space of a compact and connected Riemann surface under a holomorphic map, if it is more than one point, is an algebraic curve and the Riemann surface can be rediscovered up to an automorphism by normalizing the algebraic curve. Normalization has a very algebraic definition, a local section of the structure sheaf of the normalization is nothing but a local section of the sheaf of endomorphisms of a torsion-free coherent rank one sheaf over the original scheme (in this case the algebraic curve), and also, as was proved by Nobile, if we resolve the indeterminacy of the Gaussian map, sending each regular point of the curve to its projective tangent line, and then repeat using the suitably modified ambient manifold, we reach the normalization in finitely many steps. And within the rational function field of an irreducible algebraic variety (in this case our algebraic curve), each discrete valuation is an 'ideal point' corresponding to a prime divisor of the normalization.

At each stage of Nash's process, in arbitrary dimensions, the smooth locus is a smooth quasi-projective variety. It is worth thinking about why such a question should matter to anyone.

Mandelbrot insisted that things in mathematics which are nothing like manifolds may equally represent what is really there, such as a coastline [2].

In hard science, students trying to learn quantum theory are told that there is a chain of prerequisite definitions. The ground state component of Hydrogen, initially imagined as belonging to the real one-dimensional vector-space spanned by the function e^{-r} in suitable units of measurement, needed to be reimagined as belonging instead to the two-dimensional representation of SU_2 which only exists over the complex numbers. If someone said, "It's one of Hamilton's quaternions, then," it would lead to incorrectly believing in things like entanglement.

After Mumford left Mathematics to think collaboratively about digital technology, he had written that he had loved making mappings, had thought of it as a way to explore real things [3].

When someone says that all of reality is a manifold (or a singular manifold or a scheme, or a stack, or an etale site) they are failing to see the tentative and provisional nature which mathematical definitions must have.

Witten writes in *physics and geometry* "Spacetime is a pseudo-Riemannian manifold M , endowed with a metric tensor" [4].

Subsequent thinking of Mori and Reid, restricted specifically to quasiprojective complex varieties, says that a smooth complex variety has a minimal model with *canonical* singularities, which is locally a canonical model and more natural than a smooth model [5].

Our thinking is doomed if we try to first make an axiomatic map of possible worlds, and then find ‘nature’ as an element on our map. The tragedy is that we have no other choice if we are going to choose and enact, or decline to enact, policies. We must rely on a concept of truth. We must make definitions. Clarity of the definitions implies axioms and consistency, and a mathematical map.

The necessary consequence is that people have to rely on the mathematics which is most broadly accepted, like facts about the real number line.

I am going not going to include a political discourse, or describe the true depth of the tragedy, which can only be understood through evolutionary psychology, and through seeing in every example how things like the internet cannot couch coherent thought; but only to say that focussing on global warming does incorrectly make it seem right to install agriculture and housing on ancient wilderness nature, as long as we compensate by planting massive numbers of seedling trees.

Scientific articles which appear nowadays say that worrying about the accumulations in nature of mutations chosen by people correlates with lacking scientific understanding. That correct scientific understanding, in turn, correlates statistically with believing that each next generation of corporate, government, and academic scientists in every country know how to mutate nature based on their best goals, strategies and biological models [6].

And yet, just as in quantum theory there was a dawning realization that there is no ‘algebra’ like a number line which makes up an underlying universe, and that nor do Hydrogen waveforms really have a quaternionic component, there is also in biology the notion that there aren’t primarily ‘genes’ and ‘reading frames’ which, in the analagously trivial way, would form any sensible ambient structure pre-existing our attempt to imagine it.

The best biologists know how to give only a hint of an idea, of the type of saying, things you never thought could be related to each other, are related in surprising and direct ways, but infinitely complex and previously unknown ways. That, if you have made a permanent mutation without understanding *this*, you would have caused a tragedy.

It doesn't mean that we can call a halt to the production, and enact legislation about *this*. It means that our understanding is and always will be nothing but provisional.

The effective refutation on that front arises from a surprising place: not from advanced mathematical theory, but from the GCSE curriculum for children. Children are being encouraged to wonder: while there may be 'the gene to increase fatty acid in pork,' whether there is not going to be ever found 'the gene which make an orchid resemble a bee,' or 'the gene which controls symbiosis.'

People who live in cities, or in established countries, an increasing majority, who are said to have been 'lifted' out of poverty, have a trivialized and almost mathematical understanding of nature. They are like competent gardeners, and things are dismissed as not important in the way that gardeners dismiss weeds, or call grasshoppers 'locusts.' They are the ones who will outnumber everyone and prevail, and they will in all sincerity take a medical point of view. Like how the puffins on the Isle of Lundy are 'protected' until Lundy is a puffin farm.

Chasing the effect of repeatedly resolving the indeterminacy of *n*-forms to see where it would lead could also just be doing also the wrong thing. Mumford is right to think that just because we can make a map, doesn't in any way imply that we can rely on the map to tell us what is really there. I worry – selfishly maybe – that if mathematicians stop making *new* maps, then the last ones which were made will become trusted as if they were the only ones there, and thus had been a *necessary part* of existence.

Whereas also, people who live in new, or not yet 'developed'

countries, like America was not long ago, while they have a very deep understanding of nature, one which can never be put into words, and while they are horrified by anyone who thinks of development as anything but an irreversible loss, yet, they see nature as abundant, almost infinite, and forgiving.

Nash was an American, and if it is true, as people say, that he wanted to construct a limiting object out of resolving indeterminacy of n -forms, then it would be wanting to ‘follow the yellow brick road.’ That it doesn’t matter where it will go, but it will be to a new and wonderful place which will make us happy. It doesn’t matter what our actions are, the results will be consistent and wonderful. It is like the remaining people who still have ‘frontier’ optimism, in Alaska, or, I should say, now, rather, in Northern Alaska.

Subsequent to Nash’s question, Hironaka did answer the question of resolving singularities (algebraically in characteristic zero) [7]. His proof was based on reducing local multiplicities iteratively and the particular resolution depended on a choice of embedding into a smooth manifold.

Perez and Teissier [8] found an earlier reference [9] to Nash’s question; most references say it is a question of Nash, predating Hironaka’s successful proof, with Nash putting nothing in writing. Notable mathematicians have at times informally made announcements of papers which (like mine about arc lifting) have never materialized.

M. Spivakovsky has said, during a conversation in Grenoble [10], that his and Hironaka’s theorem can be understood as a modification of Nash’s question in which the surface is normalized at each step.

The same day, S. Abyankhar [11] answered a question about finding a local singularity unaffected (up to isomorphism and further localization) by a Nash transform without saying anything, but, to my best recollection of it, acting like trying to snatch a mosquito out of the air where there was no mosquito. To a later question, what are the conditions for a Nash transform to yield a normal manifold, he waited a significant time, and only said, it is a deep question [12].

Gonzales-Sprinberg [13], and later Atanasov, Lopez, Perry, Proudfoot and Thaddeus have analyzed toric case [14].

I'll also reference some things which I have not finished reading about, Encinas and Villamayor [15], Włodarczyk [16], Kollar [17], McQuillan [18], and Abramovich, Temkin and Włodarczyk [19] have removed dependence on embedding, though the later work needs stacks.

Referees while remaining anonymous have excused long delays in replying to my submissions with saying they had been working themselves on Nash's question, and in their hints to me have tended to introduce a differential geometry feeling to the subject.

Primary Decomposition

One way of understanding cycles is to view the structure sheaf of a closed subscheme (or analytic subspace) as a coherent sheaf on the ambient variety. Let's restrict to the case of schemes; in generalizing to the analytic setting we'll need to remember that a subsheaf of a coherent sheaf need not be coherent (while the kernel of a map of coherent sheaves always is).

A coherent sheaf \mathcal{D} on a quasiprojective scheme is 'coprimary' (generalizing the definition given in 'Algebre Local, Multiplicities' [20]) if there is a prime ideal sheaf \mathcal{P} so that if there is any prime ideal sheaf \mathcal{Q} and invertible sheaf \mathcal{L} and embedding $\mathcal{L} \otimes \mathcal{O}_M/\mathcal{Q} \rightarrow \mathcal{C}$ then $\mathcal{P} = \mathcal{Q}$.

In any filtration of such a \mathcal{C} with successive quotients of type $\mathcal{L} \otimes \mathcal{O}_M/\mathcal{Q}$, the prime ideal sheaf \mathcal{P} is the unique minimal element in the set of \mathcal{Q} . The occurrence of \mathcal{L} in the definition is needed so that we can be sure that \mathcal{C} does actually have such a filtration.

Within any coherent sheaf \mathcal{C} we may choose a maximal coherent subsheaf which is a sum (necessarily a direct sum) of coprimary sheaves, and I would guess that if we tensor this with the cartesian product of the local rings corresponding to the corresponding primes, the tensor product of the subsheaf should be isomorphic with the maximal Artinian subsheaf of the tensor product of all of \mathcal{C}

If there are no inclusions among these primes, we can use instead the semi-localization, and the maximal artinian subsheaf is the whole of the semilocalization of \mathcal{C} .

We also can use the completions instead of localizations, and the maximal Artinian module of the tensor product of our coherent cartesian product of the localizations is already a complete module for the cartesian product of the completions. It is complete for each cartesian factor a finite power of the radical acts by zero.

Our maximal direct sum of coprimary subsheaves of \mathcal{C} ought to be *essential* in the sense that it intersects any nontrivial subsheaf.

In this sense, subsheaves should be detected by discrete valuations. The intersection of a subsheaf with our chosen \mathcal{P} -coprimary subsheaf has a filtration with successive quotients only of the type $\mathcal{L} \otimes \mathcal{O}_M/\mathcal{Q}$ for \mathcal{Q} containing \mathcal{P} , and we may count the occurrences of \mathcal{P} directly, or just tensor the intersection with the maximal coprimary subsheaf with the finite cartesian product of localizations, and count the irreducible composition factors of the resulting Artinian module, or, equivalently, just consider the class of that module in the Grothendieck group of the cartesian product of the localizations modulo any sufficiently high power of the radical.

We cannot assemble this information into a homomorphism of Grothendieck groups, because extracting the maximal Artinian submodule of a module is not an exact functor. This is how primary decomposition departs from the theory of algebraic cycles even on smooth quasiprojective varieties.

In that special case, we can resolve the structure sheaf of a closed subscheme by a finite sequence of locally free coherent sheaves. Although the cohomology of the constant sheaf \mathcal{Z} could have torsion one can use characteristic classes to recover the equivalence class of a subscheme up to torsion and linear equivalence from the isomorphism types of the terms of the resolution. Let's discuss this.

Algebraic Cycles

The cohomology and algebraic cycles of smooth quasi-projective varieties are closely related to each other. Just as, in algebraic topology, because of the Thom collapse, we know that two smooth real manifolds are cobordant if and only if the Stiefel-Whitney numbers agree, it is also true that if two divisors on a smooth complex projective variety are linearly equivalent, they must have the same Chern numbers.

In the appendix to the Borel-Serre paper on Riemann-Roch, Grothendieck deduced a converse of this in every dimension, that two algebraic i cycles for any value of i (not only $i = n - 1$) are linearly equivalent modulo torsion on an n dimensional smooth quasi-projective manifold if and only if all the Chern classes (of the of the structure sheaves viewed as coherent sheaves on the smooth manifold) are the same, and that it is equivalent to the difference between the elements in the Grothendieck group being a torsion element plus a cycle of lower dimension.

There, the structure sheaf of a sub-variety is resolved by section sheaves of vector-bundles, and the Chern classes of a positive vector bundle are defined to be the scheme of vanishing of each number of global sections.

Lipman's question/conjecture about Chern cycles

The cohomology types of the Chern subvarieties come from Bruhat cells. While their cohomology classes, the Chern classes of vector bundles, are understood that way algebraically, there remain deep mysteries about what they actually look like geometrically. Even if M is complex affine space of dimension n and V is its tangent bundle, the scheme where vector-fields v_0, \dots, v_s fail to be linearly independent, has irreducible components all of dimension at least s . The deep and unsolved question of Lipman, formulated when he was a student of Oscar Zariski, determines whether there is actually a component of dimension at least s in every algebraic and non-smooth leaf closure, within the locus where the v_i are involutive, meaning $[v_i, v_j] = \sum a_k^{i,j} v_k$. And it seems likely that algebraicity is not relevant for the question.

The smooth locus is where the subsheaf of \mathcal{O}_M^{s+1} spanned by the action of directional derivative in each coordinate $(v_0(f), \dots, v_s(f))$ for $f \in \mathcal{O}_M$ local sections, is locally free in its own right. This just describes the points where we can enlarge the span of v_0, \dots, v_s by a local meromorphic change of basis to make them independent anyway.

The total Chern class and cohomological definition

The so-called total Chern class is the sum of the Chern classes, and it transforms direct sums of vector bundles into cup products of cohomology classes.

For a finite set of points of the Riemann sphere, we choose a rational function f sending these to a point ∞ of the Riemann sphere, and there is a complementary set of points sent to another point 0 . If we separate the two sets by a contour C , the integral $\int_C \frac{df}{f}$ can be interpreted as $2\pi i$ times the number of poles on one side, or zeros on the other.

The one-forms which have poles matching $\frac{df}{f}$ and no zeroes form a torsor, described by the additive Čech cocycle which is $\frac{df}{f}$ restricted to a neighbourhood of the contour C , and the same integral is an isomorphism invariant for this torsor. If we call the finite set with its subscheme structure S , it is also the transform along $\mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_S$ of the extension class of the Poincaré residue sequence $0 \rightarrow \Omega_{\mathbb{P}^1} \rightarrow \Omega_{\mathbb{P}^1}(\log C) \rightarrow \mathcal{O}_S \rightarrow 0$ in $Ext^1(\mathcal{O}_S, \Omega_{\mathbb{P}^1})$.

Chern character

To define a Chern character of an algebraic cycle in this setting it suffices to define the Chern character of a locally free sheaf, and by ignoring torsion nothing else is lost tensoring cohomology with \mathbb{Q} and defining Chern classes in the usual way as vanishing of tuples of sections (after twisting by a line bundle) and defining the Chern character as a rational polynomial function of the Chern classes. The resulting cohomology class in degree i for a locally free coherent sheaf \mathcal{F} if we bypass Chern classes altogether, doesn't require any denominators in its description, nor does it require reducing modulo torsion; it is merely the one represented by the Hochschild i -cocycle of $(\mathcal{F}, \mathcal{F} \otimes \Omega_M)$ sending local sections x_1, \dots, x_i to the function sending local sections f of \mathcal{F} to $f \otimes dx_1 \wedge \dots \wedge dx_i$.

The passage from first Chern classes to Chern characters is by splicing resolutions, and there have been two attempts I know of to generalize to the singular case. One thing a person could do is consider non-exact sequences of locally free sheaves (the theory of derived categories).

Here, such 'chain complexes' are locally objects that correspond to homotopy types of semisimplicial modules (by the Dold correspondence [21]). However it isn't clear why that homotopy type ought to reflect meaningful properties of an algebraic cycle.

The same cohomology class (of type $(1, 1)$) is the extension class of the exact sequence $0 \rightarrow L \otimes \Omega \rightarrow \mathcal{P}(L) \rightarrow L \rightarrow 0$ twisted by L^{-1} and it is induced from the extension class of the Poincare residue sequence itself $0 \rightarrow \Omega \rightarrow \Omega(\log D) \rightarrow \mathcal{O}_D \rightarrow 0$ by the surjection $\mathcal{O}_M \rightarrow \mathcal{O}_D$. The class has a homological definition as does the Chern character, which therefore can be defined directly and 'integrally' without needing recourse to the historically earlier notion of Chern classes of vector bundles. Even integrally, Grothendieck's appendix applies unchanged, although torsion in the Grothendieck group is not detected by Chern character with coefficients in \mathbb{Q} , nor is it correctly detected if we use constant coefficients \mathbb{Z} or any other constant coefficients.

Homotopy theory

One might question whether the techniques of resolutions and cohomology are best; in the theory of derived categories, one interprets a resolution by locally free sheaves as not only a formal alternating sum in the Grothendieck group, but one remembers the chain homotopy type of the resolution. This is more information as I mentioned; very precisely, on each affine part it captures the homotopy type of a corresponding semisimplicial module by the Dold correspondence. However it isn't clear why that homotopy type ought to reflect meaningful properties of an algebraic cycle.

The notion of considering the structure sheaf of a closed subscheme as a coherent sheaf on an ambient variety and resolving it applies when things are nonsingular; if the ambient variety is singular even when one can still find the necessary resolutions, or when one allows non-exact resolutions, the particular step where a Koszul resolution of the conormal sheaf contributes the denominator of a Todd class would need to be modified since the free resolution would not be a Koszul resolution.

An attempt to reconcile Chern classes with the singular setting may require considering that long exact sequences resemble filtrations; in homotopy theory one can to some extent reassemble long exact sequences into exact couples, and this was useful for determining homotopy types, especially stable homotopy types. In the theory of coherent sheaves, just passing to long exact sequences captures for filtrations only successive pairwise extensions and a little more than that, but misses the re-assembling that takes place in a more correct analogy with homotopy theory. The difficulty is caused by the fact that we define coherent sheaves by keeping track of sections over open sets rather than constructible sets. The denominator in the Todd genus comes from an assumption of local complete intersection, for example, and to get past that one ought to be able to iterate Poincare residues when nesting is more general; this requires considering sections on constructible sets which are not necessarily open. We will not deal with that here.

Valuations

In conclusion, extending the theory of algebraic cycles past the smooth case requires abandoning the Todd genus in Riemann-Roch, whose denominator is the class of the exterior algebra of the conormal sheaf of an algebraic cycle. It requires abandoning the particular formula for the Todd genus, since the exterior algebra of a conormal sheaf need not be resolving in the non-smooth case. It requires considering that chain complexes of coherent sheaves only capture something like the abelianization of a homotopy type, and finally that coherent sheaves themselves as objects of a category have a serious limitation when one looks at examples of trying to recapture algebraic cycles from their Chern classes, that one would need to consider sections over constructible sets rather than only over open sets.

A theory of valuations is a first attempt to rescue what one can in the singular case.

Integral closure

If we make no distinction between saying f is a local section of \mathcal{F} versus $f\mathcal{G}$ agrees with a subsheaf of $\mathcal{F}\mathcal{G}$, the tensor product modulo torsion, for \mathcal{G} torsion-free of rank one, on an irreducible scheme, then we are saying that we make no distinction between local sections of \mathcal{F} versus local sections of its ‘integral closure.’

If we *need not* make any such distinction, for \mathcal{F} the structure sheaf, we are saying rational maps to \mathbb{P}^1 have no a meromorphic section of a line bundle defined in codimension two extends to a legitimate meromorphic section.

Review of construction of \mathbb{R}

One way to construct the reals \mathbb{R} is as contravariant functors $\mathbb{Q} \rightarrow \mathit{Sets}$, an example of the now old construction which one can find in [22], and works with the \mathbb{Q} as a partially-ordered set replaced by any small category. According to Yoneda's lemma We identify a rational number α with the representable functor, the closed interval $[-, \alpha]$ leaving the first argument unspecified. Then for example the colimit of $[-, \alpha]$ over α such that $\alpha^2 < 2$ is the functor $[-, \sqrt{2})$ which assigns to any rational number β the rational points in the interval $(\beta, \sqrt{2})$.

A group completion

For a reduced and irreducible scheme M let's make a definition which surely must pre-date Weil divisors; it agrees with the Weil divisor group for smooth projective curves, but in general is larger.

We begin with the standard notion that a *fractional ideal sheaf* is a torsion-free coherent integrally closed rank one sheaf \mathcal{I} labelled by a nontrivial map (embedding) $\mathcal{I} \rightarrow K$ of quasi-coherent sheaves to the constant sheaf of rational functions K . The fractional ideal sheaves have a binary operation of multiplication which sends \mathcal{I} and \mathcal{J} to $\mathcal{I} \otimes \mathcal{J} / \text{torsion}$ labelled with the product labelling. It is the natural image of $\mathcal{I} \otimes \mathcal{J}$ under the tensor product of the two embeddings. However, for our purposes we need a different binary operation on only the set of integrally closed fractional ideals.

We say a fractional ideal sheaf \mathcal{I} is *integrally closed* if there is no strictly larger fractional ideal sheaf \mathcal{I}' with fractional ideal \mathcal{J} such that $\mathcal{J}\mathcal{I} = \mathcal{J}\mathcal{I}'$.

Now we introduce a binary operation that is different than just product of fractional ideal sheaves

Lemma. The binary operation on integrally-closed fractional ideal sheaves assigning to \mathcal{I} and \mathcal{J} the integral closure of $\mathcal{I}\mathcal{J}$ is associative and satisfies cancellation.

Proof. The only thing needing proof is the cancellation rule. Suppose that $\mathcal{I}\mathcal{J}$ and $\mathcal{I}\mathcal{L}$ have the same integral closure where \mathcal{J} and \mathcal{L} are integrally closed. Then there is some \mathcal{M} so that $\mathcal{M}\mathcal{I}\mathcal{J} = \mathcal{M}\mathcal{I}\mathcal{L}$ and so taking the product with $\mathcal{M}\mathcal{I}$ makes \mathcal{L} and \mathcal{J} become equal, by definition of integral closure they are already equal.

It follows that

Corollary. The monoid of integrally closed fractional ideal sheaves with this binary operation embeds faithfully in to its group completion Γ .

We denote by $\nu(\mathcal{I})$ the image of the integrally closed fractional ideal \mathcal{I} in this group. Although the letter ν corresponds linguistically to the roman n , its visual similarity for v means it has been used traditionally to stand for ‘valuation.’

Also, if \mathcal{I} is not integrally closed, we also write $\nu(\mathcal{I})$ for the image in Γ of the integral closure of \mathcal{I} . Thus we think of ν as a sort-of universal valuation, taking values in the commutative group Γ .

‘Rational’ fractional ideals

Next, instead of choosing a character $\nu : \Gamma \rightarrow \mathbb{Q}$ which might describe a particular valuation, we should merely consider the natural $\Gamma \cong \Gamma \otimes \mathbb{Z} \rightarrow \Gamma \otimes \mathbb{Q}$ induced by $\mathbb{Z} \subset \mathbb{Q}$. By a slight abuse of notation we’ll also denote by $\nu(\mathcal{I})$ the image of (the integral closure of) a fractional ideal \mathcal{I} in $\Gamma \otimes \mathbb{Q}$, however, when we do that, it is no longer true that ν faithfully represents integrally closed fractional ideals, as the map $\Gamma \rightarrow \Gamma \otimes \mathbb{Q}$ will have nontrivial kernel when Γ has torsion.

Let’s say that an element $\gamma \in \Gamma \otimes \mathbb{Q}$ belongs to the *effective cone* if there is a nonzero natural number n such that $n\gamma$ is the image of an element in the effective cone of Γ .

There is a partial ordering on $\Gamma \otimes \mathbb{Q}$ such that $\nu(\mathcal{I}) \leq \nu(\mathcal{J})$ if and only if after modifying \mathcal{I} or \mathcal{J} by addition of a torsion element, there are representatives of \mathcal{I}' and \mathcal{J}' in Γ and a number N so that (the integral closure of) \mathcal{J}'^N is contained in the integral closure of \mathcal{I}'^N . (Note that the inequality is reversed from the familiar one that relates inclusions of sets with their cardinalities).

Thus we think of ν as a sort-of universal valuation, taking values in the commutative group $\Gamma \otimes \mathbb{Q}$.

Completion of the group-completion

Finally, we are going to complete $\Gamma \otimes \mathbb{Q}$ in the other sense. The completion of $\Gamma \otimes \mathbb{Q}$ can be defined using representable functors, as we did for \mathbb{Q} itself already, I expect that the result will be no different than $\Gamma \otimes \mathbb{R}$. In that case we are now working in a real vector-space $\Gamma \otimes \mathbb{R}$ with an effective cone.

Nash transforms

In the case of iterating Nash transforms, one has a sequence of varieties $\dots V_2 \rightarrow V_1 \rightarrow V_0 = V$ and if we call Ω_{ij} the pullback modulo torsion to V_i of the one-forms of V_j , then taking n 'th exterior power and again reducing modulo torsion we obtain a system of locally free sheaves, which we can write $\mathcal{O}_{V_i}(K_j)$ with K_j a Cartier divisor on V_j for $i > j$.

One thing which we will do later, as naturally as possible, is to restrict to a subvariety of V which supports n generically independent vector-fields, and to describe $\mathcal{O}_{V_i}(K_j)$ for $j < i$ as an ascending series of locally principal ideal sheaves on V_i using the theory of connections. On the union of the regular locus $U_0 \rightarrow U_1 \rightarrow U_2 \dots$ there is then an infinite ascending chain of ideal sheaves, and the colimit is a locally principal ideal sheaf on the whole of the Nash manifold U .

The ideal sheaves are not 'relatively basepoint-free' however when V_0 and therefore all V_i are irreducible, any fractional ideal sheaf on any V_i can be described as the tensor difference of two ideal sheaves on V_0 . A fractional ideal sheaf is just a torsion-free rank-one sheaf labelled by an embedding in the constant sheaf of rational functions, and the same is true more simply if we ignore labellings, that any coherent sheaf on any V_i is the result of pulling back from V_0 a torsion free coherent sheaf whose reduction modulo torsion becomes invertible, and tensoring the inverse with the pullback of another coherent sheaf on V_0 , tensoring and again reducing modulo torsion.

In this sense, the i 'th locally principal ideal sheaf on U in our series is the result of pulling back $\Lambda^{n+1}\mathcal{P}(\mathcal{F})$, tensoring with the inverse of the pullback modulo torsion of \mathcal{F}^{r+1} and again reducing modulo torsion. And here \mathcal{F} must be defined recursively, to be the product of the first, second, third, ... $(i-1)$ 'st locally principal ideal sheaves.

Because of naturality of first principal parts, we may apply \mathcal{P} at any stage on or past the i 'th stage. In some sense the most efficient way is to apply it always at the i 'th stage, and then the result is not yet invertible, all the earlier ones are, and in fact the $i + 1$ 'st Nash transform is the universal one which makes this invertible.

A natural transformation

It is convenient to leave the torsion-free rank one sheaf \mathcal{F} unspecified, and then there is a natural map (natural transformation of functors)

$$\Lambda^{n+1}\mathcal{P}(\mathcal{F})^{\otimes n+2} \rightarrow \Lambda^{n+1}\mathcal{P}(\mathcal{F} \otimes \Lambda^{n+1}(\mathcal{P}(\mathcal{F}))) \quad (1)$$

which one can reduce modulo torsion. It is analgous to ‘carrying’ when adding base $n + 2$ expansions. If we write $\mathcal{F}_0 = \mathcal{O}_M$, $\mathcal{F}_1 = \Lambda^{n+1}\mathcal{P}(\mathcal{F})/torsion$, $\mathcal{F}_2 = \mathcal{F}_1^{\otimes 2}/torsion$, ... $\mathcal{F}_{n+1} = \mathcal{F}_1^{\otimes(n+1)}/torsion$, $\mathcal{F}_{n+2} = \Lambda^{n+1}\mathcal{P}(\mathcal{F}\mathcal{F}_1)/torsion$, $\mathcal{F}_{n+3} = \mathcal{F}_{n+2}\mathcal{F}_1/torsion$, ... and in general $\mathcal{F}_{(n+2)^{i+1}} = \Lambda^{n+1}\mathcal{P}(\mathcal{F}\mathcal{F}_1\mathcal{F}_{(n+2)}\dots\mathcal{F}_{(n+2)^i})$, then from the maps $\mathcal{F}_i^{\otimes n+2} \rightarrow \mathcal{F}_{i(n+2)}$ we can derive existence of particular natural maps $\mathcal{F}_a \otimes \mathcal{F}_b \rightarrow \mathcal{F}_{a+b}/torsion$. The \mathcal{F}_i are torsion-free and rank one, and are generated by global sections if \mathcal{F} is very ample.

In [23] in an initial result, I showed that in characteistic zero the necessary and sufficient condition for Nash’s process to finish in finitely many steps is that starting with $\mathcal{F} = \mathcal{O}_M$, eventually $\mathcal{F}_{(n+2)^i}^{\otimes n+2}/torsion \rightarrow \mathcal{F}_{(n+2)^{i+1}}/torsion$ should become an isomorphism after multiplying by a power of \mathcal{F}_i and \mathcal{F}_{i+1} .

The argument in [24] using finiteness of normalization improves this, showing that in fact that the following simpler condition is necessary and sufficient, that eventually. $\mathcal{F}_{(n+2)^i}^{\otimes n+2} \rightarrow \mathcal{F}_{(n+2)^{i+1}}$ is surjective (equivalently, an isomorphism).

Therefore, a necessary and sufficient condition for the existence of a torsion-free coherent rank one resolving sheaf \mathcal{F} (whose existence is equivalent to Hironaka’s theorem) is that on the first stage

$$\Lambda^n\mathcal{P}(\mathcal{F})^{\otimes(n+2)}/torsion \rightarrow \Lambda^{n+1}\mathcal{P}(\mathcal{F} \otimes \Lambda^{n+1}(\mathcal{P}(\mathcal{F})))$$

Here is how the proof in [24] works. The weaker statement is equivalent where we allow ourselves to multiply by a power of $\mathcal{F}\Lambda^{n+1}\mathcal{P}(\mathcal{F})/torsion$. The way to prove that multiplying by

this power is unnecessary is to assemble $\mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \dots$ into a sheaf of graded rings on V_0 . The indexing of the \mathcal{F}_i reflects this grading which wasn't evident in earlier notation.

Here is how we prove that surjectivity after multiplying by a power of $\mathcal{F} \wedge^{n+1} \mathcal{P}(\mathcal{F})$ implies we can find a new choice of \mathcal{F} where surjectivity holds without needing to multiply by a power like that. The combinatorics of adding base $n+2$ expressions for natural numbers imply that finite type of the graded sheaf of rings is equivalent to one of the stronger equalities holding, with \mathcal{F} replaced by some $\mathcal{F}_1 \mathcal{F}_{(n+2)} \dots \mathcal{F}_{(n+2)^i} = \mathcal{F}_{\frac{(n+2)^{i+1}-1}{n+1}}$, whereas the weaker condition where we allow multiplying by the power only implies that the normalization of the graded sheaf of rings is finite type.

We deduce the stronger equality from the weaker one, for a possibly larger value of i , as follows: Let's describe the affine case so the language of sheaves doesn't get in the way. The subring of the graded ring generated up to each degree i is finite type and has a normalization which is finitely generated as a module over this ring. The series of normalizations is an increasing series of subrings of the overall normalization, which exhausts the overall normalization, then contains all of the finite number of ring generators of that. It follows that the normalization of one of the finite type subrings is the same as the overall normalization. But the intermediate subrings between these are an ascending chain of submodules of a finitely-generated module over a Noetherian ring. This completes the proof.

Discussion

While there are no natural inclusions among the \mathcal{F}_i , there are natural maps $\mathcal{F}_{(n+2)^i}^{\otimes n+2}/torsion \rightarrow \mathcal{F}_{(n+2)^{i+1}}$. When $\mathcal{F} = \mathcal{O}_M$ then \mathcal{F}_i is naturally a subsheaf of the rational function field tensor $\Lambda^n \Omega_M^{\otimes i}$. If one wishes a premonition of our arguments using connections, one can merely calculate the \mathcal{F}_i when we have embedded Ω_M into a free module using a sequence of generically independent global derivations, and see that inside the fraction field tensor $\Lambda^n \Omega_M^{\otimes i}$ lives inside of \mathcal{O}_M .

Coherent sheaves make sense analytically, and finding an \mathcal{F} which makes (1) surjective comprises a second order differential equation and we've just shown that solving it is equivalent to resolving the singularities of M . The equivalence, ever, being indirect.

Let's re-describe what we've just done in a more geometric language. Once we resolve the indeterminacies of \mathcal{F} to obtain a $W_0 \rightarrow M_0$, where the pullback modulo torsion of \mathcal{F} is invertible, and then if we further resolve indeterminacies of n -forms on W_0 to obtain $W_1 \rightarrow W_0$ the pullback of $\mathcal{F} \otimes \Lambda^{n+1} \mathcal{P}(\mathcal{F})$ modulo torsion becomes an invertible sheaf, is the one which we were calling $\mathcal{F}\mathcal{F}_1$ above, with the multiplication being tensor product modulo torsion, and our map is

$$\mathcal{F}_1^{\otimes n+2} \rightarrow \Lambda^{n+1} \mathcal{P}(\mathcal{F}\mathcal{F}_1).$$

Tensoring with the $-n-1$ tensor power of \mathcal{F}_1 then gives

$$\mathcal{F}_1 \rightarrow \mathcal{F}_1^{\otimes -n-1} \Lambda^{n+1} \mathcal{P}(\mathcal{F}\mathcal{F}_1)$$

and \mathcal{F}_1 itself has a similar form to the right side of the display; if we write the pullback modulo torsion of \mathcal{F} itself as \mathcal{I} it is $\mathcal{I}^{\otimes -n-1} \Lambda^{n+1} \mathcal{P}(\mathcal{I})$. So we are looking at a map

$$\mathcal{I}^{\otimes -n-1} \Lambda^{n+1} \mathcal{P}(\mathcal{I}) \rightarrow (\mathcal{F}\mathcal{F}_1)^{\otimes -n-1} \Lambda^{n+1} \mathcal{P}(\mathcal{F}\mathcal{F}_1).$$

Both the domain and codomain are expressed as a multiplicative difference of relatively basepoint-free coherent sheaves, and this map, on W_1 , is precisely the one which determines

ramification; it is the map from the pullback modulo torsion of the n -forms of W_1 to the n -forms modulo torsion of W_2 .

If the *original* map (1) is surjective, this one is too, and $W_1 \rightarrow W_0$ is an unramified Nash transformation. Note that this is not quite the same as saying that W_1 resolves the singularities of M_0 . However, W_1 is nonsingular, because we are in characteristic zero, or working analytically, and the map now expresses an isomorphism between the n -forms modulo torsion and a locally free sheaf.

Conversely, if \mathcal{F} is a resolving sheaf, W_0 is smooth, and (1) becomes surjective after pulling back and reducing modulo torsion. This only implies that (2) becomes surjective after tensoring with a tensor power of $\mathcal{F} \otimes \Lambda^{n+1}\mathcal{P}(\mathcal{F})$, however, as we showed above, that if we repeatedly replace \mathcal{F} by $\mathcal{F} \otimes \Lambda^{n+1}\mathcal{P}(\mathcal{F})$ and the *just once* apply $\Lambda^{n+1}\mathcal{P}(-)$ then when we replace \mathcal{F} by the new sheaf, the map (1) has become surjective. I here since it uses Hilbert's basis theorem.

It is interesting that all we have been talking about so far is whether a second order differential equation can characterise resolutions, and it has led us into needing to consider the same algebraic structure which relates to Nash's question.

The discussion generalizes if we wanted to talk about either algebraic cycles and cohomology, or indeterminacy of coherent rank one sheaves, however let's focus here on the resolution problem.

Moduli of resolutions

The isomorphism types of torsion-free coherent \mathcal{F} which make (1) surjective comprise a type of infinite-dimensional moduli space. Although the relation with resolving is indirect (on the one hand, requiring us to replace \mathcal{F} by some higher \mathcal{F}_i if we wish to literally obtain a resolving sheaf, and conversely requiring us to replace the purported resolution by a Nash blowup we wish it to literally correspond to one of the \mathcal{F}), a way of approaching the question of finding resolutions within a finite-dimensional moduli space would be to filter moduli of solutions of the surjectivity of (1) by finite-dimensional spaces.

Connections

Let's suppose we're given a connection

$$\nabla : \mathcal{L} \rightarrow \mathcal{L} \otimes \Omega$$

satisfying

$$\nabla(fy) = f\nabla(y) + y \otimes df$$

for local sections y of \mathcal{L} and f of \mathcal{O}_M .

The fibre vectoriel L whose structure sheaf pushes down on M to $\mathcal{O}_M \oplus \mathcal{L} \oplus \dots$ has one-forms

$$(\mathcal{O}_M d\mathcal{O}_M) \oplus (\mathcal{L}d\mathcal{O}_M + \mathcal{O}_M d'\mathcal{L}) \oplus \dots$$

Here d' is the differential on the line bundle, and when we take forms with logarithmic poles on the zero section and twist by the zero section given multiplicity -1 , The 'degree-one' term which survives is $\mathcal{P}(\mathcal{L})$. The first summand is a subsheaf isomorphic with $\mathcal{L} \otimes \Omega_M \oplus \mathcal{L}$, while the second summand is only a coherent subsheaf after we reduce modulo the first term, since d is not \mathcal{O}_M linear. However, the set of local sections $-\nabla(y) + 1 \otimes dy$ does describe a subsheaf, as for y a local section of \mathcal{L} we have $-\nabla(fy) + d(fy) = -f\nabla(y) - y \otimes df + y \otimes df + f \otimes dy = f \cdot (-\nabla(y) + dy)$ for local sections f of \mathcal{O}_M showing it is stable under multiplication by local sections of \mathcal{O}_M .

Locally, once we choose a generating section x of \mathcal{L} then for each f we have

$$\nabla(yf) = y\nabla(f) \oplus f \otimes dy.$$

In fact, $\mathcal{L}^{-1}\mathcal{F}$ is an ideal sheaf, and y ranges over local sections of this ideal sheaf, and we have the element $\nabla(f)$ depending on our choice of generator, and locally $\mathcal{P}(\mathcal{F})$ is isomorphic with the subsheaf of $\mathcal{F} \oplus \Omega$ spanned by $y \oplus dy$. The isomorphism depends on the choice of f , and it sends $y \oplus dy$ to $y\nabla(f) \oplus f \otimes dy$.

Therefore we could define $\mathcal{P}(\mathcal{F})$ by a type of additive cocycle. If we call the ideal sheaf \mathcal{I} first we construct on each open set

where f is a local section the subsheaf of $\mathcal{I} \oplus \Omega$ spanned by the $y \oplus dy$.

Covering M by open sets, the patching Čech cocycle for subsheaf described in the summand on the right side is easy to describe, it is just the ordinary patching in $\mathcal{F} \otimes \Omega$. On an open set where f is a chosen generator of \mathcal{L} and another where g is a chosen generator, the patching isomorphism is multiplication by the unit $u = g^{-1}f$.

For the patching of the whole sheaf, when $g = uf$ as before, each $\nabla(f)$ is equal to

$$\begin{aligned} \nabla(ug) &= u\nabla(g) \oplus g \otimes du \\ &= f g^{-1}\nabla(g) + g \otimes (gdf - fdg)/g^2 \\ &= f g^{-1}\nabla(g) + df - f \frac{dg}{g} \end{aligned}$$

The nicest way of writing this is in terms of $\nabla \log(g) = \frac{1}{f}\lambda(f)$, where it says

$$\nabla \log(f) - \nabla \log(g) = d \log(f) - d \log(g).$$

Instead of saying that $\mathcal{P}(\mathcal{F})$ is spanned by the $y\nabla(f) \oplus f \otimes dy$ we could have said that it is spanned by the $yf(\nabla(\log f) \oplus 1 \otimes d \log(y))$.

If we multiply f by a local unit and divide y by the same, if we call this unit s , then $\nabla(\log f)$ becomes $\frac{1}{sf}\nabla(sf) = \frac{1}{sf}(s\nabla(f) + f \otimes ds) = \nabla \log(f) + d \log(s)$ while $d \log(y)$ becomes $d \log(y) - d \log(s)$ and the expression is unaffected.

Therefore the element $(\nabla \log f \oplus 1 \otimes d \log y)$ is a well-defined rational section of first principal parts depending only on the product yf , and if we had named the spanning element with the symbol $\nabla'(yf)$ with ∇' denoting what we would think of as a universal connection, we would be factorizing $\nabla'(yf) = yf\nabla' \log yf$.

The universal connection ∇' can likely be understood directly by thinking of first principal parts related to the square of the ideal defining the diagonal.

In an earlier explicit description we said when we have generically independent derivations $\delta_1, \dots, \delta_n$ with values in the rational function field K then we define $\mathcal{P}(\mathcal{I})$ for \mathcal{I} an ideal sheaf, or fractional ideal sheaf, to be spanned by local sections $y \oplus \delta_1(y) \oplus \dots \oplus \delta_n(y)$ in $\mathcal{F} \oplus K^n$ for y local sections of the ideal sheaf.

Recall that $(\nabla \log f \oplus 1 \otimes d \log y)$ depends only on yf for f a local section of \mathcal{L} , and when $f = 1$ this is just $1 \otimes d \log y$. So the product with y becomes $y \oplus dy$, and we see that the description of the universal connection agrees with what we had said when we described principal parts using derivations.

Earlier I wrote that if \mathcal{I} is spanned locally by sections t_i and \mathcal{O} is spanned by x_i then letting s_i be the products $t_j x_k \Lambda^{n+1} \mathcal{P}(\mathcal{I}) / \text{torsion}$ is spanned by $(s_0 \oplus \nabla s_0) \wedge \dots \wedge (s_n \oplus \nabla(s_n))$ where $1 \oplus \nabla$ is the 'universal connection,' and this calculated to $s_0 \dots s_n d \log(s_1/s_0) \wedge \dots \wedge d \log(s_n/s_0)$.

By $1 \oplus \nabla$ I really meant that we can calculate the universal and invariant connection ∇' in terms of our ordinary connection ∇ using our more rigorous formula $\nabla' \log yf = \nabla \log f + d \log y$ with ∇ our actual connection

Thus there is a theoretical way of describing the original construction using derivations as coming from a choice of connection, but in any case when \mathcal{F} is an ideal sheaf and δ_i derivations of \mathcal{O}_M generically independent and well-defined on a neighbourhood we can use that to embed $\mathcal{P}(\mathcal{F})$ into \mathcal{O}_M^{n+1} .

For some reason we wanted to take the δ_i commuting.

We always have

$$\mathcal{F}_{(n+2)^\alpha}^{n+2} \subset \mathcal{F}_{(n+2)^{\alpha+1}} \subset (\mathcal{F} \mathcal{F}_1 \dots \mathcal{F}_{(n+2)^\alpha})^{n+1}$$

and if I replace the first factor \mathcal{F}^{n+1} on the right with \mathcal{F}_1 we can compose the chain of inclusions $\mathcal{F}_1^{n+2} (\mathcal{F}_{(n+2)} \dots \mathcal{F}_{(n+2)^\alpha})^{r+1} \subset$

$$\mathcal{F}_{(n+2)}^{n+2}(\mathcal{F}_{(n+2)^2} \dots \mathcal{F}_{(n+2)^\alpha})^{r+1} \subset \dots \subset \mathcal{F}_{(n+2)^\alpha}^{r+2}.$$

Now recall that for a fractional ideal \mathcal{I} we denote by $\nu(I)$ the associated element of $\Gamma \otimes \mathbb{R}$

We are going to apply this to the chain of inclusions we've established

$$\begin{aligned} & \dots \frac{1}{(n+2)^\alpha} \nu(\mathcal{F}_{(n+2)^\alpha}) \geq \frac{1}{(n+2)^{\alpha+1}} \nu(\mathcal{F}_{(n+1)^{\alpha+1}}) \geq \dots \\ & \dots \geq \frac{n+1}{(n+2)^{\alpha+1}} \nu(\mathcal{F}\mathcal{F}_1\mathcal{F}_{(n+2)} \dots \mathcal{F}_{(n+2)^\alpha}) \geq \frac{n+1}{(n+2)^\alpha} \nu(\mathcal{F}\mathcal{F}_1 \dots \mathcal{F}_{(n+2)^{\alpha+1}}) \\ & \geq \frac{1}{(n+2)^\alpha} \nu(\mathcal{F}_{(n+2)^\alpha}) - \frac{n+1}{(n+2)^\alpha} \nu(\mathcal{F}) + \frac{1}{(n+2)^\alpha} \nu(\mathcal{F}_1). \end{aligned}$$

I'd like to explain the somewhat unusual use of ellipses ... here. Except for the last term written one sees in infinite increasing chain followed by an infinite decreasing chain of larger numbers. The last inequality written then shows that each term of the decreasing chain of larger numbers can be made less than corresponding element of the smaller increasing chain by subtracting a rational multiple of the constant $\nu(\mathcal{F}_1) - (n+1)\nu(\mathcal{F})$, and the coefficients in that rational multiple tend to zero.

Each fractional ideal is a functor of \mathcal{F} of a particular degree, and the denominator we've introduced as a coefficient to each ν value is the degree. That is to say, if we abbreviate by $\nu(\mathcal{F})$ the expression $\frac{1}{\text{degree}(\mathcal{F})} \nu(\mathcal{F})$ then our inequalities become

$$\begin{aligned} & \dots \mu(\mathcal{F}_{(n+2)^\alpha}) \geq \mu(\mathcal{F}_{(n+1)^{\alpha+1}}) \geq \dots \\ & \dots \geq (n+1) \mu(\mathcal{F}\mathcal{F}_1\mathcal{F}_{(n+2)} \dots \mathcal{F}_{(n+2)^\alpha}) \geq (n+1) \mu(\mathcal{F}\mathcal{F}_1 \dots \mathcal{F}_{(n+2)^{\alpha+1}}) \\ & \geq \mu(\mathcal{F}_{(n+2)^\alpha}) - \frac{n+1}{(n+2)^\alpha} \nu(\mathcal{F}) + \frac{1}{(n+2)^\alpha} \nu(\mathcal{F}_1). \end{aligned}$$

It follows that the $\mu(\mathcal{F}_{(n+2)^\alpha})$ form a decreasing sequence, and the $(n+1)\mu(\mathcal{F}\mathcal{F}_1 \dots \mathcal{F}_{(n+2)^\alpha})$ comprise an increasing sequence, and

Theorem. Provided a suitable connection exists (i.e. locally on M) then the limit of the increasing sequence in the completion of $\Gamma \otimes \mathbb{Q}$ is equal to the limit of the decreasing sequence.

The fact that the limit of the sequence of interest, the increasing sequence, is equal to the limit of the decreasing sequence means we may change focus and try to understand the decreasing sequence.

Although, the definitions of the two sequences are intertwined in a particular recursive way, also.

Pre-conclusion

My thinking on these matters is not independent. M. Thaddeus, in a letter to Teissier about the preprint with Perez, said that a purported open cover of a toric variety actually covers only the rational points. The thinking above is partly generated by this idea. For a toric variety, a rational character of the lattice corresponds to an equivariant rational curve, and the question whether minimum point of the image of our ascending sequence stabilizes is equivalent to lifting the arc to the smooth locus of the Nash manifold. One visualizes the arc lifted part-by-part extending across an infinite union of open subsets exhausting the Nash manifold, and whether the open cover is finite.

Although Thaddeus didn't specify what he means by a rational point, earlier, speaking generally, M. Spivakvski said [10] that valuations ought to be viewed as generalizations of arcs, and arcs can be viewed as homomorphisms to a formal power series ring.

My construction of Γ and of a completion of $\Gamma \otimes \mathbb{Q}$ is meant to be able to speak of a type of non-rational point which I think that both conversations were referring to in different ways. If we manage to skip the step of tensoring with \mathbb{Q} the completion might have torsion. We will have a toric interlude, and the last of this paper will establish the conditions mentioned in [26] necessary and sufficient to lift a *holomorphic* arc, although, our results above and in the next section suggest that lifting of arcs which go to ideal points of M (in the sense that the imaginary axis goes to a cusp in a modular curve), would also be necessary to prove properness of Nash's manifold.

Toric interlude

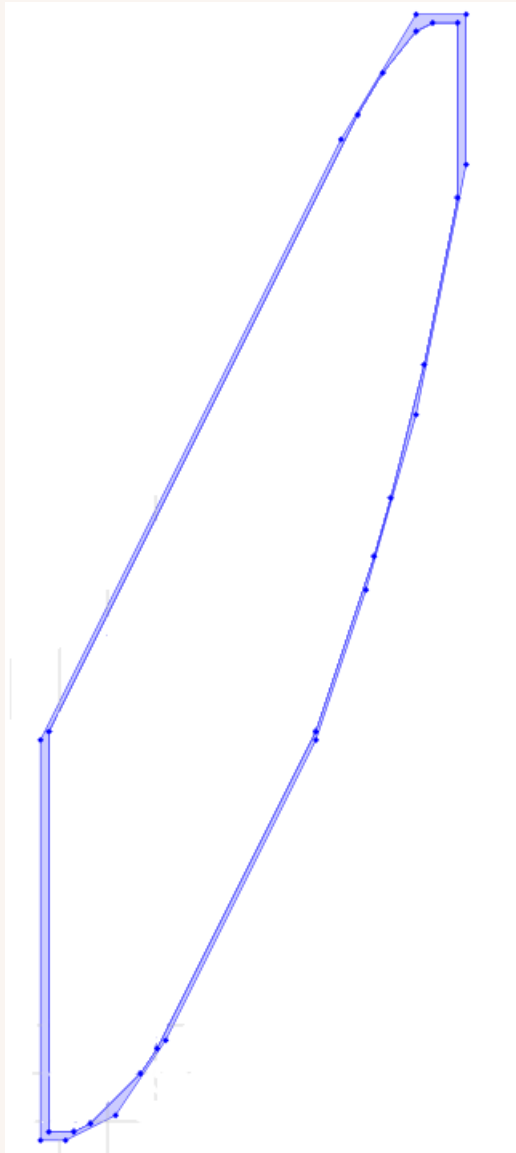
When M is a projective toric variety and Λ is the lattice of characters, once \mathcal{F} is a very ample invertible sheaf generated by a finite set of torus characters all \mathcal{F}_i are also generated by a finite set of torus characters. The $\mu(\mathcal{F}_i)$ can be understood as corresponding to particular convex subsets of either $\Lambda \otimes \mathbb{Z}[\frac{1}{n+2}]$ or more simply, just $\Lambda \otimes \mathbb{Q}$. Each $\mu(\mathcal{F}_{(n+2)^{i+1}})$ corresponds precisely to the convex hull of the set of averages of $n+1$ global sections of $\mathcal{F}_{\frac{(n+2)^{i+1}-1}{n+1}} = \mathcal{F}\mathcal{F}_1 \dots \mathcal{F}_{(n+2)^i}$ which are not on a hyperplane in $\Lambda \otimes \mathbb{Q}$ while each $\mathcal{F}\mathcal{F}_1 \dots \mathcal{F}_{(n+2)^i}$ corresponds to the convex hull of the result of iterating averages of the larger number of $(n+2)$ elements not on a hyperplane, iterated i times. Note in both cases we do not pass to convex hull until after the iterated averaging is done.

A character $\Lambda \rightarrow \mathbb{R}$ can be evaluated on each $\mu(\mathcal{F}_i) \in \Gamma \otimes \mathbb{R}$ by evaluating on the corresponding subset of $\Lambda \otimes \mathbb{Q}$ and choosing the minimum value.

At the same time, characters $\Lambda \rightarrow \mathbb{R}$ correspond to arcs in the toric variety. Characters which factorize through \mathbb{Q} correspond to equivariant rational curves, while those which do not correspond to a type of non-holomorphic arc. In both cases liftability to the smooth locus is determined by whether the increasing sequence has finitely many distinct values in the image.

If our basic torus characters are x_1, \dots, x_n then a rational character determines in a primitive generator a primitive rational monomial in the x_i , while a real character can determine a function of the type $x_i(t) = e^{\lambda_i t}$ with λ_i positive real, which converge not as $t \rightarrow 0$ but as $t \rightarrow -\infty$ along the real line.

Example When the equivariant global sections of \mathcal{F} correspond to $\{(0, 0), (0, 1), (1, 0), (1, 3), (2, 5)\} \subset \mathbb{Z}^2$ the figure on the next page shows convex hull of $\frac{1}{16}$ times the lattice point set $\{(5, 5), (5, 21), \dots\}$ corresponding to \mathcal{F}_{16} and the convex hull of $\frac{1}{48}$ times the lattice point set $\{(16, 16), (16, 64), \dots\}$ corresponding to \mathcal{F}_{48} . The theorem of the section above called ‘Connectoins’ implies that as either of the powers q, s is increased, the set-theoretic difference of $\frac{1}{4^q}$ times the hull of the lattice set of \mathcal{F}_{4^q} minus $\frac{1}{3 \cdot 4^s}$ times the hull of the lattice set of $\mathcal{F}_{3 \cdot 4^s}$ decreases monotonically or stays constant, and as $q, s \rightarrow \infty$ the difference shrinks to a one-dimensional continuous curve of area zero. For general projective toric varieties the theorem implies that it will always be a topological sphere. This is consistent with the stronger notion that the limiting curve is the boundary of a polyhedron. For projective toric varieties, the stronger fact will hold precisely when the inner curve stabilizes in finitely many steps (the outer curve never does, but always converges to the limit of the inner curve). The stronger assertion remains unproven and would be strictly stronger if and only if the arc lifting problem for such exponential types of arcs, converging as $t \rightarrow -\infty$, were inequivalent to the arc lifting problem for holomorphic arcs, of the type converging when $t \rightarrow 0$.



The javascript for graphing the convex hulls on a canvas is due to [indy356.github.com](https://github.com/indy356).

Proof of the theorem of [26].

Let M be a singular normal irreducible complex manifold of dimension n . Let $f : U \rightarrow M$ be the smooth Nash manifold, which is the smooth locus of the inverse image of the tower of Nash transforms of M . Let $\gamma : R \rightarrow M$ be an analytic map from a connected Riemann surface whose image contains a nonsingular point of M .

Conjecturally, it is sometimes said, there is a unique $\gamma' : R \rightarrow U$ such that $\gamma = f \circ \gamma'$.

The Nash manifold U supports Cartier divisors K_0, K_1, \dots so that the coherent sheaf $\mathcal{O}_U(K_i)$ is the pullback modulo torsion of the differential n -forms of the i 'th Nash transform; these are allowed to have infinitely many irreducible components. By a triangular change of basis, we let

$$D_{(n+2)^j} = K_j + (n+1) \sum_{t=0}^{j-1} (r+2)^{j-1-t} K_j$$

(we will define D_j for non-powers of $n+2$ a bit later). The reason for doing that is that the divisors D_j are relatively basepoint-free; in fact we can give an elementary description of a coherent sheaf \mathcal{F}_j on M which pulls back modulo torsion to $\mathcal{O}_U(D_j)$.

One way of remembering the formulas above is to solve for K_U as an infinite series

$$\frac{-1}{(n+1)} K_U = D_1 + D_{(n+2)} + D_{(n+2)^2} + \dots$$

The left side of the equation describes an effective Cartier divisor now, albeit a \mathbb{Q} -divisor. If our arc is unbounded the tails of the series would be allowed to get smaller as we pass from consideration of one singular point to another, and would be allowed to intersect trivially.

The purpose of this note is to explain a theorem which is stated in the earlier preprint, which in this language says

Theorem. Let $\gamma : R \rightarrow M$ be an analytic map from a connected Riemann surface whose image contains just one non-singular point of M . Then there is a lift $\gamma' : R \rightarrow U$ such that $\gamma = f \circ \gamma'$ if and only if the series of intersection numbers

$$\gamma \cdot D_{n+2} + \gamma \cdot D_{(n+2)^2} + \gamma \cdot D_{(n+2)^3} \dots$$

is eventually (necessarily) nonzero to the base of $(n + 2)$.

The theorem will extend to the case when the image of γ contains finitely many singular points; however, if γ is unbounded it is possible that the tails for the separate singular points might intersect trivially; in that case we could apply the theorem to each singular point to deduce arc lifting even while the entire series would not need to be geometric.

If we start with an arc γ which is already lifted to U , the case when the series converges literally is when it has only finitely many nonzero terms. In this case, the arc maps to a single point of M ; it always will share its fiber with a complete curve of the other type.

When the condition is holds with infinitely many nonzero terms, and γ meeting only one singular point, it is even in this case not only an abstract condition. The formula for the sum of a geometric series remains valid, and it is allowed to be interpreted formally or p -adically for a prime divisor of $n + 2$.

The proof of the theorem as we've stated it here is a formal manipulation. A bit later we will describe generators of coherent ideal sheaves on M itself which pull back modulo torsion to the $\mathcal{O}_U(D_i)$. The proof is just formal, and all we need to know is that all are effective. The formal proof is to rewrite the right side as

$$\frac{1}{n+1}(-D_1 + ((n+2)D_1 - D_{(n+2)}) + ((n+2)D_{(n+2)} - D_{(n+2)^2}) + \dots)$$

Multiplying both sides by $n + 1$ gives

$$-K_U = -D_1 + ((n+2)D_1 - D_{(n+2)}) + ((n+2)D_{(n+2)} - D_{(n+2)^2}) + \dots$$

Each term in round parentheses is one of the effective divisors $-(K_{i+1} - K_i)$

It does not matter whether we view these as divisors on the Nash manifold itself or, term-by-term, each on a sufficiently high Nash blowup that they become Cartier. This expresses an effective Cartier divisor $-K_U$ as another Cartier divisor $-D_1$ minus effective divisors. D_1 is the extension of K_M across U , starting with the smooth locus of M , obtained by pulling back $\Lambda^n \Omega_M$ along $U \rightarrow M$.

The hypothetical limiting natural number $-K_U \cdot \gamma$ is the Lefschetz product $-D_1 \cdot \gamma$ plus a sum of negative numbers, which must be finite. If we insist on thinking in terms of intersection numbers, we have to allow some of the finitely many components P of $| -D_1 |$ containing our smooth point of $\gamma(R)$ to contain the whole of it under all linear equivalences which keep $| -K_U |$ effective; the same integer multiple of each $P \cdot \gamma$ occurs on both sides of the equation and it does not matter what value we should assign to it.

It is much easier to say that we are talking about a very particular finite ascending chain of locally principal ideals defined on finite terms in the tower of Nash transformations, which we can pull back along successive lifts of γ to obtain a finite chain of locally free ideal sheaves. It is just a combinatorial fact that the series in the theorem is a geometric series when this series for $-K_U$ actually does have finitely many nonzero terms.

The principle which we're applying at this point is just a general principle. Beyond considering curves, if we consider any irreducible subvariety V of M containing a smooth point, we can define V to have a 'stable formal proper transform' if the stable locus of proper transforms of V in finite chains of Nash transforms is proper over V . Then the theorem generalizes to say that once V has a stable formal proper transform, the necessary and sufficient condition to actually have a proper transform in U is that the ascending series of divisors $K_0 + (K_1 - K_0) + (K_2 - K_1) + \dots$ converges in the literal sense once restricted to V . The geometric series involving our relatively baspoint-free divisors D_i reduces combinatorially to this easy general principle, in other words.

In either case, what we have then is an inverse system of line bundles which restricts to a finite inverse system, and has as its limit the intersection.

Let's represent the limiting bundle as the restriction of the canonical bundle of U along an actual morphism. A rough argument is that on any bounded open set, the effective Cartier discrepancy

$$K_{i+1} - K_i = D_{(n+2)^{i+1}} - (n+2)D_{(n+2)^i}$$

must restrict to a principal divisor for sufficiently large i , which can be moved away from the stable proper transform of V .

This is true, but a more careful consideration shows that when the divisor restricts to a principal divisor, it actually only implies that the Nash blowups are finite maps, at points of the transform of V within that bounded neighbourhood. Consideration of the analytic local rings of Nash blowups of M at such points of the stable proper transform, and the theorem of finiteness of normalization, implies then that each bounded open subset of the stable proper transform is eventually contained in the smooth locus of a finite Nash blowup, and therefore contained in an open subset of U , which is the union of these.

This finishes the proof of theorem 1 under the assumption $-K_U$ is effective. Next, one can argue as follows: if we consider any point $p \in M$, there is a neighbourhood of p in M whose smooth Nash manifold has an effective anticanonical divisor. This can be seen various ways. One way to think of it is that however singular the point p may be, we can find n flows which are everywhere defined in a neighbourhood W of p and transverse where they cross some other point q in that neighbourhood.

For a moment use the letter U to denote the Nash manifold of W . Since $U \rightarrow W$ is natural the flows lift to flows on U . Evaluating on the wedge product of the corresponding global vector fields on U gives a map $\mathcal{O}_U(K_U) \rightarrow \mathcal{O}_U$ showing that $-K_U$ is effective.

Let's explicate the geometry which allowed this to take place. Thus let's consider the particular \mathcal{F}_i defined earlier which pull back modulo torsion to $\mathcal{O}(D_i)$. The tuple of vector fields on W gives a map

$$\Omega_W/torsion \rightarrow B,$$

where B is nothing but the section sheaf of a trivial bundle of rank n . The deRham differential gives a connection

$$d : \mathcal{O}_W \rightarrow \mathcal{O}_W \otimes B$$

of course.

Starting from $\mathcal{F} = \mathcal{O}_W$, the sequence of sheaves

$$\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \dots$$

now each has a corresponding holomorphic connection

$$\nabla : \mathcal{F}_i \rightarrow \mathcal{F}_i \otimes B$$

and the sheaf of n forms mod torsion in the i 'th Nash blowup $\pi : W' \rightarrow W$ is

$$(\pi^* \mathcal{F}^{n+1} \pi^* \mathcal{F}_{(n+2)^{i-1}}/torsion)^{-1} \pi^* \mathcal{F}_{(n+2)^i}/torsion.$$

In more detail, let

$$I = \mathcal{F}_1 \mathcal{F}_{(n+2)} \dots \mathcal{F}_{(n+2)^{i-1}}.$$

Then

$$\mathcal{F}_{(n+2)^{i-1}} = I^{n+1},$$

$\mathcal{F}_{(n+2)^i}$ is defined to be $\Lambda^{n+1} \mathcal{P}(\mathcal{F}I)/torsion$, and the n forms on the i 'th Nash blowup are

$$(\pi^* \mathcal{F} \pi^* I/torsion)^{-n-1} \pi^* \Lambda^{n+1} \mathcal{P}(\mathcal{F}I)/torsion, \quad (1)$$

in this case with $\mathcal{F} = \mathcal{O}_W$

Once t_i are local sections which span I and x_i are local coordinates generating \mathcal{O}_W then letting the s_i be the products $t_j x_k$ we have that $\Lambda^{n+1} \mathcal{P}(I)/torsion$ is spanned by the

$$(s_0 \oplus \nabla s_0) \wedge \dots \wedge (s_n \oplus \nabla s_n).$$

The connection $1 \oplus \nabla$ is a connection with values in principal parts rather than differentials, which is essentially a formal object whose existence is assured because of properties of the pullback of a coherent sheaf to its own Fibré Vectoriel.

This can be expanded in terms of a tangible connection ∇ though as

$$\begin{aligned} & \sum_{i=0}^n (-1)^i s_i \nabla(s_0) \wedge \dots \wedge \widehat{\nabla s_i} \wedge \dots \wedge \nabla s_n \quad (2) \\ &= s_0 \dots s_n \sum_{i=0}^n (-1)^i \nabla \log(s_0) \wedge \dots \wedge \widehat{\nabla \log s_i} \wedge \dots \wedge \nabla \log s_n \end{aligned}$$

where the hat denotes a deleted term.

For any i, j we have

$$\begin{aligned} \nabla(s_j) &= \nabla\left(s_i \frac{s_j}{s_i}\right) \\ &= \frac{s_j}{s_i} \nabla(s_i) + s_i \otimes d\left(\frac{s_j}{s_i}\right) \end{aligned}$$

and so

$$\nabla \log s_j - \nabla \log s_i = d \log\left(\frac{s_j}{s_i}\right).$$

Using this, eliminate ∇ from the answer to obtain an expression which is just one term, not a sum and not involving ∇

$$s_0 \dots s_n \cdot d \log(s_1/s_0) \wedge \dots \wedge d \log(s_n/s_0).$$

This shows that the answer does not depend on ∇ ; we are allowed to choose it.

When i is not an $(n+2)$ power but has an expansion

$$\sum a_j (n+2)^j$$

with $0 \leq a_j < (n+2)$ \mathcal{F}_i is defined to be $\prod \mathcal{F}_{(n+2)^j}^{a_j}$. Thus all \mathcal{F}_i contain \mathcal{F}_1^i and any connection ∇ on \mathcal{F}_1 extends to a meromorphic connection

$$\mathcal{F}_i \xrightarrow{\nabla} \mathcal{F}_i \otimes \Omega_W.$$

However, the connection which is constructed explicitly using a basis of local derivations (which we can even take to be commuting) is a holomorphic connection

$$\mathcal{F}_i \rightarrow \mathcal{F}_i \otimes B$$

for all i . The formula (2) is linear of degree $n+1$ with respect to multiplying the s_i by a meromorphic function, as the later calculation explains. By the product rule the connection gives

$$\mathcal{F}I \rightarrow \mathcal{F}I \otimes B$$

and the coefficients in $\mathcal{F}I$ which is locally principal once pulled back to W' pass through (2) and cancel in (1) leaving

$$\Lambda^n \Omega_{W'} / \text{torsion} \rightarrow \Lambda^n (\mathcal{O}_{W'} \otimes B) \cong \mathcal{O}_{W'}.$$

The transition map

$$\begin{aligned} & (\tau^*(\mathcal{F}\mathcal{F}_1 \dots \mathcal{F}_{(n+2)^{i-1}}) / \text{torsion})^{-n-1} \tau^* \mathcal{F}_{(n+2)^i} / \text{torsion} \\ & \rightarrow (\tau^*(\mathcal{F}\mathcal{F}_1 \dots \mathcal{F}_{(n+2)^i}) / \text{torsion})^{-n-1} \tau^* \mathcal{F}_{(n+2)^{i+1}} / \text{torsion}. \end{aligned}$$

for the Nash blowup $\tau : W'' \rightarrow M$ of W' is the evident rearrangement of terms in

$$(\mathcal{F}\mathcal{F}_1 \dots \mathcal{F}_{(n+2)^i})^{n+1} \mathcal{F}_{(n+2)^i} \rightarrow (\mathcal{F}\mathcal{F}_1 \dots \mathcal{F}_{(n+2)^{i-1}})^{n+1} \mathcal{F}_{(n+2)^{i+1}}.$$

induced by carrying $\mathcal{F}_{(n+2)^i}^{n+2} \rightarrow \mathcal{F}_{(n+2)^{i+1}}$.

The inclusion commutes with this transition map; and in the limit on U it embeds $\mathcal{O}_U(K_U) \rightarrow \mathcal{O}_U$ as the coherent sheaf of ideals defining an effective anticanonical divisor. Thus if one doesn't wish to trust an argument based on symmetry alone, it is possible to prove that the anticanonical divisor of the Nash manifold is linearly equivalent to a particular effective divisor when restricted to the inverse image of sufficiently small open neighbourhoods of a point. QED

Remark. It is probably nicer to show that there is always a locally principal sheaf J on M conducting n forms into an ideal sheaf on U , analogous to how $d\log(s_1/s_0)$ has no zeroes on the projective line being a section of $\mathcal{O}(-[s_1] - [s_0])$. Maybe $\mathcal{F} \cong \mathcal{O}_W$ on open sets W are restrictions $\mathcal{F}(W)$ of some \mathcal{F} with connection on M . J might be $\Lambda^{-n}B$ for B containing the one forms of M , with the meromorphic extension $\mathcal{F}_i \rightarrow \mathcal{F}_i \otimes B$ holomorphic everywhere.

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